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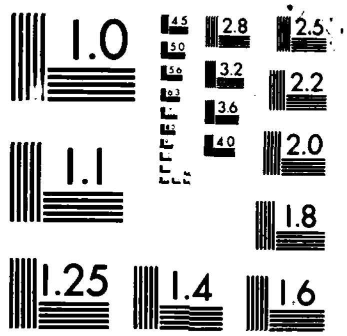
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RADC-TR-84-9, Pt V (of six)
Final Technical Report
April 1984

ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY PERFECTLY CONDUCTING BODIES MOVING IN VACUUM

A Description of Motion and Deformation Retardation of Sets and Functions

University of Delaware

Allan G. Dallas

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Air Force Systems Command
Griffiss Air Force Base, NY 13441

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The idea of a "Motion" is introduced as a mathematical framework within which to describe a fairly broad class of moving and deforming bodies. The implications of the definition are explored. In particular, the space-time track of the motion is represented in terms of the instantaneous positions of the moving body. The spatial projections of the intersections of the space-time track of a motion with characteristic cones are represented and studied. Finally, classes of smooth motions are identified. The additional properties of such motions, including the existence of a normal velocity field, are established.

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ORIENTATION

This is Part V of a six-part report on the results of an investigation into the problem of determining the scattered field resulting from the interaction of a given electromagnetic incident wave with a perfectly conducting body executing specified motion and deformation in vacuum. Part I presents the principal results of the study of the case of a general motion, while Part II contains the specialization and completion of the general reasoning in the situation in which the scattering body is stationary. Part III is devoted to the derivation of a boundary-integral-type representation for the scattered field, in a form involving scalar and vector potentials. Parts IV, V, and VI are of the nature of appendices, containing the proofs of numerous auxiliary technical assertions utilized in the first three parts. Certain of the chapters of Part I are sufficient preparation for studying each of Parts III through VI. Specifically, the entire report is organized as follows:

The section- and equation-numbering scheme is fairly self-explanatory. For example, "[I.5.4]" designates the fourth section of Chapter 5 of Part I, while "(I.5.4.1)" refers to the equation numbered (1) in that section; when the reference is made within Part I, however, these are shortened to "[5.4]" and "(5.4.1)," respectively. Note that Parts II-VI contain no chapter-subdivisions. "[IV.14]" indicates the fourteenth section of Part IV, "(IV.14.6)" the equation numbered (6) within that section; the Roman-numeral designations are never dropped in Parts II-VI.

A more detailed outline of the contents of the entire report appears in [I.1.2]. An index of notations and the bibliography are also to be found in Part I. References to the bibliography are made by citing, for example, "Mikhlin [34]." Finally, it should be pointed out that notations connected with the more common mathematical concepts are standarized for all parts of the report in [I.1.4].

PART V

A DESCRIPTION OF MOTION AND DEFORMATION. RETARDATION OF SETS AND FUNCTIONS

We concern ourselves here with a precise mathematical realization of a fairly broad class of physical bodies moving and deforming in space; of course, to be subsumed here is the possibility that the body under consideration is fixed. In so doing, we shall be specifying the types of regions in space-time in which we are studying electromagnetic scattering phenomena, as well as providing the groundwork for the introduction of certain potential functions defined and examined in Chapter [I.5] and Part IV. Also, the results of this Part V allow us to set up, in Chapter [I.4], the requisite exact classical formulation of the family of scattering problems under consideration, as initial-boundary-value problems for Maxwell's equations.

We begin with the basic definition of a "motion," and establish various geometric and topological implications of this definition. Subsequently, we take up the connections between a motion and characteristic cones. Finally, the basic classes of "smooth" motions are defined, and their important properties are described.

As always, c denotes the constant appearing in Maxwell's equations, representing the speed of light in vacuum.

[V.1] D E F I N I T I O N S. Let $\mathbb{M}$ denote the family of all ordered triples $(\{\mathcal{B}_\zeta\}_{\zeta\in\mathbb{R}}, \mathcal{R}, \chi)$, wherein

(i) for each $\zeta \in \mathbb{R}$, $\mathcal{B}_\zeta$ is a non-void, compact, and regularly closed subset of $\mathbb{R}^3$ possessing connected complement, while $\mathcal{R} \subset \mathbb{R}^3$ is non-void and compact;

(ii) χ: $\partial\mathcal{R}\times\mathbb{R} \to \mathbb{R}^3$ is a continuous function such that, for each $\zeta \in \mathbb{R}$, $\chi(\cdot,\zeta)$: $\partial\mathcal{R} \to \mathbb{R}^3$ is an injection taking $\partial\mathcal{R}$ onto $\partial\mathcal{B}_\zeta$; moreover, $\chi(P,\cdot)$ is Lipschitz continuous on $\mathbb{R}$, uniformly as P ranges over $\partial\mathcal{R}$, with

$$c^* := \sup \left\{ \frac{|\chi(P,\zeta_2)-\chi(P,\zeta_1)|_3}{|\zeta_2-\zeta_1|} \;\middle|\; P \in \partial\mathcal{R}; \quad \zeta_1,\zeta_2 \in \mathbb{R}, \; \zeta_1 \neq \zeta_2 \right\} \in [0,c); \tag{1}$$

(iii) whenever $\zeta \in \mathbb{R}$ and $(\zeta_i)_{i=1}^\infty$ is a sequence in $\mathbb{R}$ which converges to ζ, then

$$\lim_{i\to\infty} \lambda_3(\mathcal{B}_{\zeta_i} \Delta \mathcal{B}_\zeta) = 0;$$

and

(iv) whenever $\zeta \in \mathbb{R}$ and $Z \in \partial\mathcal{B}_\zeta$, there exists a positive number $\eta(Z,\zeta)$ such that for each $\varepsilon \in (0,\eta(Z,\zeta)]$, there can be found continuous functions $p_{I\varepsilon}$: $[-\varepsilon,\varepsilon] \to \mathbb{R}^3$ and $p_{E\varepsilon}$: $[-\varepsilon,\varepsilon] \to \mathbb{R}^3$ satisfying

$$\left.\begin{array}{l} p_{I\varepsilon}(\xi) \in B^3_\varepsilon(Z)^- \cap \mathcal{B}^o_{\zeta+\frac{1}{c}\xi}, \\ \text{and} \\ p_{E\varepsilon}(\xi) \in B^3_\varepsilon(Z)^- \cap \mathcal{B}'_{\zeta+\frac{1}{c}\xi}, \end{array}\right\} \text{ whenever } \quad |\xi| \leq \varepsilon. \tag{2}$$

Define an equivalence relation $\mathcal{E}$ on $\mathbb{M}$ by declaring that, if $(\{B_{i\zeta}\}_{\zeta\in\mathbb{R}}, R_i, \chi_i) \in \mathbb{M}$ for $i = 1$ and 2, then $(\{B_{1\zeta}\}_{\zeta\in\mathbb{R}}, R_1, \chi_1)\mathcal{E}(\{B_{2\zeta}\}_{\zeta\in\mathbb{R}}, R_2, \chi_2)$ iff there exists a continuous bijection $F_{12}\colon\ \partial R_1 \to \partial R_2$ such that

$$\chi_1(P,\zeta) = \chi_2(F_{12}(P),\zeta) \quad \text{for each} \quad P \in \partial R_1, \quad \zeta \in \mathbb{R}. \tag{3}$$

Then a *motion* is an equivalence class of elements of $\mathbb{M}$ under the equivalence relation $\mathcal{E}$.

Let $M \subset \mathbb{M}$ be a motion, and $(\{B_\zeta\}_{\zeta\in\mathbb{R}}, R, \chi) \in M$: R is termed a *reference set*, χ a *motion function*, and (R,χ) a *reference pair* for the motion M. A property or quantity associated with $(\{B_\zeta\}_{\zeta\in\mathbb{R}}, R, \chi)$ is said to be *intrinsic to* M iff it is possessed by or characteristic of every element of M. ■.

[V.2] R E M A R K S. We shall maintain here the notation of [V.1].

(a) It is known that the collection of all subsets of $\mathbb{R}^3$ with finite Lebesgue measure is a (complete) metric space when equipped with the metric given by $(A,B) \mapsto \lambda_3(A\Delta B)$, with the understanding that sets A and B for which $\lambda_3(A\Delta B) = 0$ are to be identified (so that, more precisely, we must consider the corresponding collection of equivalence classes); cf., *e.g.*, Hewitt and Stromberg [20]. Now, Postulate [V.1.iii] says that the map $\zeta \mapsto B_\zeta$ on $\mathbb{R}$ into this metric space is continuous, a most reasonable requirement from the standpoint of intuition. Condition [V.1.iii] is used only in the proof of the extremely important "intermediate-value," or "boundary-crossing,"

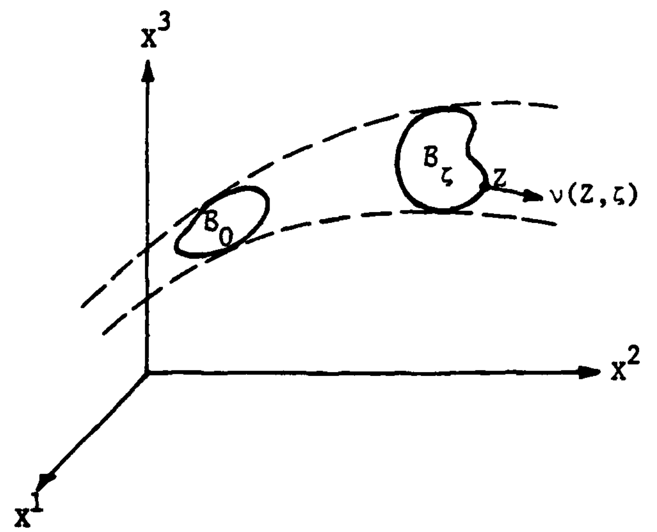

FIGURE 1. Body moving and deforming in space

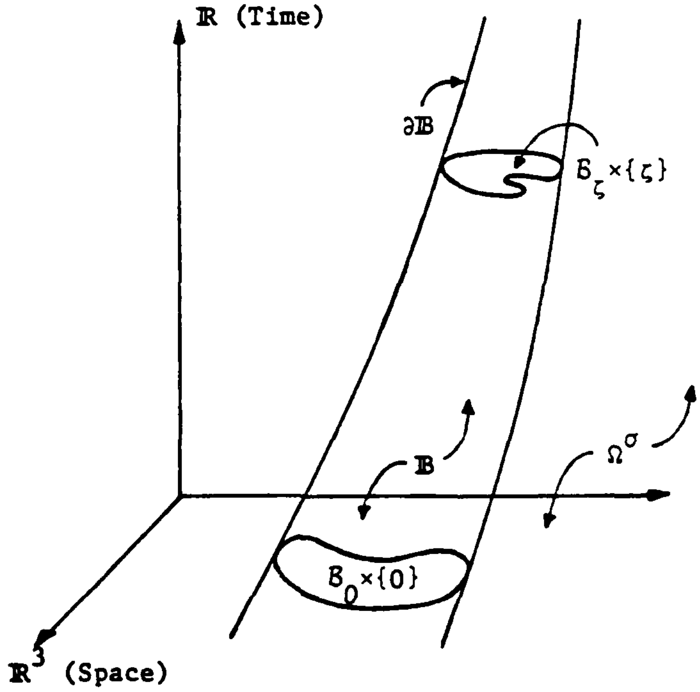

FIGURE 2. Body moving and deforming in space-time

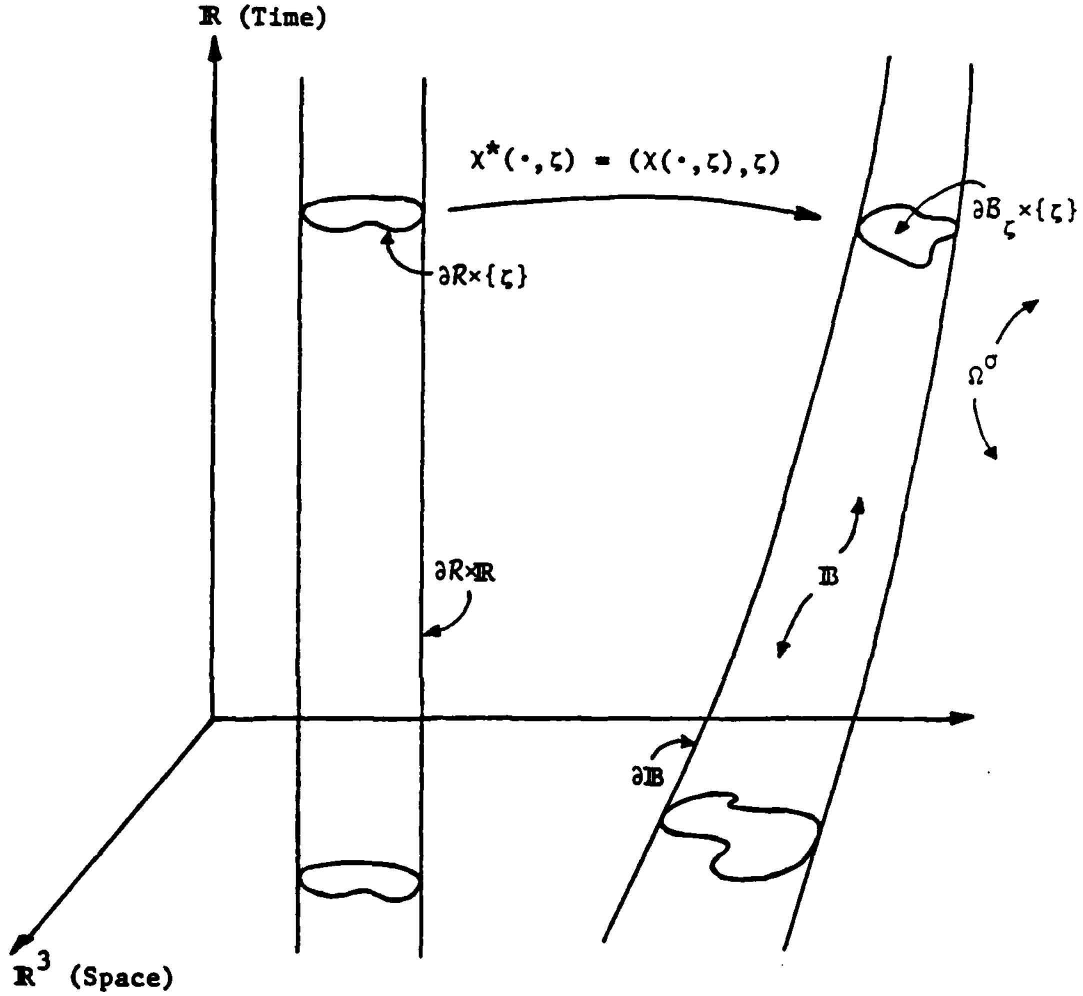

FIGURE 3. Definition of "motion"

Lemma [V.9], *infra*. If the latter can be proven by using only [V.1.i and ii] (which we have not succeeded in doing, but strongly suspect is possible), a concomitant economization of [V.1] would be in order, through the deletion of [V.1.iii].

(b) The status of condition [V.1.iv] is not unlike that of [V.1.iii]: we have found [V.1.iv] to be a sufficient condition under which the urgently needed statement of Proposition [V.27], *infra*, can be verified; if, as we believe to be entirely possible, [V.27] can be substantiated solely on the basis of [V.1.i and ii], then it would be permissible to do away with Postulate [V.1.iv]. In fact, it appears reasonable to expect that [V.1.iv] follows from [V.1.i and ii], in which case we could, of course, certainly eliminate the former from [V.1]. It shall follow from the upcoming Lemma [V.12], which is proven without recourse to [V.1.iv], that whenever $\zeta \in \mathbb{R}$, $Z \in \partial B_\zeta$, and $\varepsilon > 0$, then

$$\left.\begin{aligned} & B^3_\varepsilon(Z) \cap B^o_{\zeta + \frac{1}{c}\xi} \neq \emptyset, \\ \text{and} \quad & \\ & B^3_\varepsilon(Z) \cap B'_{\zeta + \frac{1}{c}\xi} \neq \emptyset, \end{aligned}\right\} \text{ for } |\xi| \leq \varepsilon.$$

Thus, the axiom of choice can be invoked to conclude that there exist functions on $[-\varepsilon,\varepsilon]$ into $\mathbb{R}^3$ for which (V.1.2) holds; the important additional assertion of [V.1.iv] is that *continuous* functions of this sort can be found, for all sufficiently small $\varepsilon > 0$ (depending on Z and ζ).

In any event, we must point out that [V.1.iv] is a fairly weak restriction on the collection $\{\mathcal{B}_\zeta\}_{\zeta\in\mathbb{R}}$, and is satisfied in most cases of practical interest. In particular, it is fulfilled when each $\partial\mathcal{B}_\zeta$ is a (2,3;1)-manifold, and even when some or all of the sets $\{\partial\mathcal{B}_\zeta\}_{\zeta\in\mathbb{R}}$ exhibit corners and spikes, provided these are neither "too severe" nor "too closely spaced." On the other hand, it is easy to generate examples in which each $\partial\mathcal{B}_\zeta$ possesses very severe corners, spikes, and cusps, but for which [V.1.iv] holds. For more details concerning this matter, one should consult [V.28-30].

(c) If the number c^* of (V.1.1) is equal to zero, then conditions [V.1.iii and iv] are trivially fulfilled; cf., [V.5.c].

(d) The inequality $c^* < c$ of (V.1.1) is crucial for later developments, exerting a decisive effect on the geometry of a motion in space-time; this will become clear as we proceed. Loosely speaking, the condition demands that "the 'speed' of each 'point on the surface of the moving body' is always less than c."

(e) $\mathcal{E}$ is indeed an equivalence relation on $\mathbb{M}$: its reflexivity and transitivity are certainly easily verified, while to check its symmetry we need only observe that, with notation as in [V.1], since ∂R_1 is compact, the continuous bijection F_{12}: $\partial R_1 \to \partial R_2$ is in fact a homeomorphism, so its inverse F_{21}: $\partial R_2 \to \partial R_1$ is a continuous bijection, with $\chi_2(P,\zeta) = \chi_1(F_{21}(P),\zeta)$ for $P \in \partial R_2$ and $\zeta \in \mathbb{R}$.

(f) For each $\zeta \in \mathbb{R}$, the requirement of [V.1.i] that $\mathcal{B}_\zeta$ be regularly closed, *i.e.*, that $B_\zeta^{o-} = B_\zeta$, implies that B_ζ^o is non-void, B_ζ^o and B_ζ' are regularly open, and $\partial B_\zeta = \partial\{B_\zeta^o\} = \partial\{B_\zeta'\}$.

(g) Let $\zeta \in \mathbb{R}$. By [V.1.ii], $\chi(\cdot,\zeta)$: $\partial R \to \partial B_\zeta$ is a continuous bijection, whence the compactness of ∂R shows that this map is a homeomorphism. Thus, the collection $\{\{\partial B_s\}_{s\in\mathbb{R}}, \partial R\}$ consists of pairwise homeomorphic sets.

(h) Let M be a motion. We can already identify several intrinsic properties of M:

(i) c^*, as computed from the Definition (V.1.1), using any element $(\{B_\zeta\}_{\zeta\in\mathbb{R}}, R, \chi) \in M$
[Indeed, if also $(\{\tilde{B}_\zeta\}_{\zeta\in\mathbb{R}}, \tilde{R}, \tilde{\chi}) \in M$, and we denote the corresponding constant by $\tilde{c}^*$, then the equality $c^* = \tilde{c}^*$ is a *simple consequence* of the existence of a bijection F: $\partial R \to \partial\tilde{R}$ such that $\chi(P,\zeta) = \tilde{\chi}(F(P),\zeta)$ for each $P \in \partial R$ and $\zeta \in \mathbb{R}$, as one can readily check, with (V.1.1).],

(ii) B_ζ, for each $\zeta \in \mathbb{R}$, wherein $(\{B_s\}_{s\in\mathbb{R}}, R, \chi) \in M$
[For, suppose that $(\{\tilde{B}_s\}_{s\in\mathbb{R}}, \tilde{R}, \tilde{\chi})$ is also in M, and let F: $\partial R \to \partial\tilde{R}$ be as in (i). Choose $\zeta \in \mathbb{R}$. Then

$$\partial B_\zeta = \chi(\partial R,\zeta) = \tilde{\chi}(F(\partial R),\zeta) = \tilde{\chi}(\partial\tilde{R},\zeta) = \partial\tilde{B}_\zeta.$$

Thus, since B_ζ and $\tilde{B}_\zeta$ are closed, we have $B_\zeta = B_\zeta^o \cup \partial B_\zeta = B_\zeta^o \cup \partial\tilde{B}_\zeta$ and $\partial\tilde{B}_\zeta \subset \tilde{B}_\zeta$, so

$$B_\zeta^{-\prime} \cap B_\zeta = B_\zeta^{-\prime} \cap (B_\zeta^o \cup \partial B_\zeta^-) = B_\zeta^{-\prime} \cap B_\zeta^o ,$$

and we can write

$$B_\zeta^{-\prime} = (B_\zeta^{-\prime} \cap B_\zeta^o) \cup (B_\zeta^{-\prime} \cap B_\zeta^{\prime}).$$

From this, upon recalling that $B_\zeta^{-\prime}$ is connected ([V.1.i]), and observing that $B_\zeta^{-\prime} \cap B_\zeta^o$ and $B_\zeta^{-\prime} \cap B_\zeta^{\prime}$ are mutually separated (since this is certainly true of B_ζ^o and $B_\zeta^{\prime}$), we deduce that exactly one of the sets $B_\zeta^{-\prime} \cap B_\zeta^o$ and $B_\zeta^{-\prime} \cap B_\zeta^{\prime}$ is empty. If the latter were empty, we should find that $B_\zeta^{-\prime} \subset B_\zeta$, which is clearly impossible, since $B_\zeta^{-\prime}$ is unbounded, while B_ζ is bounded. Therefore, $B_\zeta^{-\prime} \cap B_\zeta^o = \emptyset$, whence $B_\zeta^o \subset B_\zeta^-$, and $B_\zeta = B_\zeta^{o-} \subset B_\zeta^-$. We are led to the reversed inclusion by similar reasoning, so $B_\zeta^- = B_\zeta$, and B_ζ is indeed intrinsic to M.],

and

(iii) $\cup_{P \in \partial R}\, \chi(P,\mathbb{R})$, the set swept out by the "particle paths" for M, wherein (R,χ) is any reference pair for M [This follows easily from (ii), since the set in question is just $\cup_{\zeta \in \mathbb{R}}\, \partial B_\zeta$.].

(j) Let M be a motion. Fix $\zeta \in \mathbb{R}$. There exists a reference pair for M of the form (B_ζ, χ^ζ), wherein B_ζ is the intrinsic "position" of the motion at the "time" ζ, and $\chi^\zeta(\cdot,\zeta)$ is the identity function on ∂B_ζ. To see this, let $(\{B_s\}_{s\in\mathbb{R}}, R, \chi) \in M$, and denote the inverse of $\chi(\cdot,\zeta)$: $\partial\mathcal{R} \to \partial\mathcal{B}_\zeta$ by χ_ζ^{-1}: $\partial\mathcal{B}_\zeta \to \partial\mathcal{R}$. Define χ^ζ: $\partial\mathcal{B}_\zeta\times\mathbb{R} \to \mathbb{R}^3$ by setting

$$\chi^\zeta(Y,s) := \chi(\chi_\zeta^{-1}(Y),s) \quad \text{for each} \quad Y \in \partial\mathcal{B}_\zeta, \quad s \in \mathbb{R}.$$

Clearly, $\chi^\zeta(\cdot,\zeta)$ is the identity on $\partial\mathcal{B}_\zeta$. In order to check that $(\{\mathcal{B}_s\}_{s\in\mathbb{R}}, \mathcal{B}_\zeta, \chi^\zeta) \in \mathbb{M}$, we observe first that the requirements of [V.1.i, iii, and iv] are obviously fulfilled. Concerning the satisfaction of the conditions of [V.1.ii], we begin by noting that the map $(Y,s) \mapsto (\chi_\zeta^{-1}(Y),s)$ is continuous on $\partial\mathcal{B}_\zeta\times\mathbb{R}$ onto $\partial\mathcal{R}\times\mathbb{R}$ (the coordinate functions of the map are continuous, since χ_ζ^{-1}: $\partial\mathcal{B}_\zeta \to \partial\mathcal{R}$ is continuous), while χ: $\partial\mathcal{R}\times\mathbb{R} \to \mathbb{R}^3$ is continuous, so χ^ζ is continuous, as the composition of these two maps. Next, for any $s \in \mathbb{R}$, $\chi^\zeta(\cdot,s) = \chi(\chi_\zeta^{-1}(\cdot),s)$: $\partial\mathcal{B}_\zeta \to \partial\mathcal{B}_s$ is injective, for χ_ζ^{-1} and $\chi(\cdot,s)$ are injections. Further, $\chi_\zeta^{-1}(\partial\mathcal{B}_\zeta) = \partial\mathcal{R}$, while $\chi(\partial\mathcal{R},s) = \partial\mathcal{B}_s$, so $\chi^\zeta(\partial\mathcal{B}_\zeta,s) = \partial\mathcal{B}_s$. Finally,

$$\sup\left\{\frac{|\chi^\zeta(Y,s_2)-\chi^\zeta(Y,s_1)|_3}{|s_2-s_1|}\,\middle|\; Y \in \partial\mathcal{B}_\zeta;\quad s_1,s_2 \in \mathbb{R},\quad s_1 \neq s_2\right\}$$

$$= \sup\left\{\frac{|\chi(P,s_2)-\chi(P,s_1)|_3}{|s_2-s_1|}\,\middle|\; P \in \partial\mathcal{R};\quad s_1,s_2 \in \mathbb{R},\quad s_1 \neq s_2\right\}$$

$$= c^* \in [0,c),$$

with c^* denoting the intrinsic constant for $\mathcal{M}$; the second equality follows from the definition of χ^ζ and the fact that $\chi_\zeta^{-1}(\partial\mathcal{B}_\zeta) = \partial\mathcal{R}$. We can now assert that $(\{\mathcal{B}_s\}_{s\in\mathbb{R}}, \mathcal{B}_\zeta, \chi^\zeta) \in \mathbb{M}$; there remains only the verification that this ordered triple is in $\mathcal{M}$, *i.e.*, is $\mathcal{E}$-equivalent to $(\{\mathcal{B}_s\}_{s\in\mathbb{R}}, \mathcal{R}, \chi)$. But this follows directly from the fact that $\chi_\zeta^{-1}: \partial\mathcal{B}_\zeta \to \partial\mathcal{R}$ is a continuous bijection and the very definition of χ^ζ. Thus, $(\mathcal{B}_\zeta, \chi^\zeta)$ is a reference pair for $\mathcal{M}$.

(k) As usual, let $\mathcal{M}$ be a motion, and $(\{\mathcal{B}_\zeta\}_{\zeta\in\mathbb{R}}, \mathcal{R}, \chi) \in \mathcal{M}$. We have demanded only that $\mathcal{M}$ possess a sort of "continuity," in the sense prescribed by the continuity of χ and the map $\zeta \mapsto \mathcal{B}_\zeta$ (cf., (a)); no smoothness conditions have been placed on either χ or the boundaries $\partial\mathcal{R}$ and $\{\partial\mathcal{B}_\zeta\}_{\zeta\in\mathbb{R}}$. We shall later define and study various classes of motions which do possess such smoothness, after examining certain implications of the general definition of [V.1], *i.e.*, after exploiting the basic continuity postulates as far as we can.

(l) The class of physical moving bodies which can be modelled by using our definition of "motion" is limited principally by the assumption that the boundaries $\{\partial\mathcal{B}_\zeta\}_{\zeta\in\mathbb{R}}$ are mutually homeomorphic. Thus, we can describe, for example, the simultaneous movement of a collection of bodies, but it is evident that these can be permitted neither to fuse together nor even collide, nor can any one split apart; such a deficiency may prove to be prohibitive in certain contemplated applications. On the other hand, short of breaking up or bending over to touch itself, any one body can be permitted to undergo quite severe deformations, while its movement in the rigid-body sense is unrestricted, so long as $c^* < c$.

[V.3] N O T A T I O N S. Let $\mathcal{M}$ be a motion. Throughout the sequel, the symbols $\mathcal{B}_\zeta$ (for each $\zeta \in \mathbb{R}$) and c^* shall be reserved for the intrinsic properties of $\mathcal{M}$ as in [V.2.h.i and ii], while $(\mathcal{R},\chi)$ shall always denote a reference pair for $\mathcal{M}$. For each $\zeta \in \mathbb{R}$, we shall write, whenever it is convenient to do so, $\chi_\zeta := \chi(\cdot,\zeta)$: $\partial\mathcal{R} \to \partial\mathcal{B}_\zeta$; of course, just as in Remark [V.2.j], the inverse of this homeomorphism is denoted by χ_ζ^{-1}: $\partial\mathcal{B}_\zeta \to \partial\mathcal{R}$. Thus,

$$\chi(\chi_\zeta^{-1}(Z),\zeta) = \chi_\zeta(\chi_\zeta^{-1}(Z)) = Z \quad \text{for each} \quad \zeta \in \mathbb{R}, \quad Z \in \partial\mathcal{B}_\zeta, \tag{1}$$

and

$$\chi_\zeta^{-1}(\chi(P,\zeta)) = \chi_\zeta^{-1}(\chi_\zeta(P)) = P \quad \text{for each} \quad \zeta \in \mathbb{R}, \quad P \in \partial\mathcal{R}. \tag{2}$$

Further, χ^*: $\partial\mathcal{R}\times\mathbb{R} \to \mathbb{R}^4$ is the function given by

$$\chi^*(P,\zeta) := (\chi(P,\zeta),\zeta) \quad \text{for each} \quad P \in \partial\mathcal{R}, \quad \zeta \in \mathbb{R}. \tag{3}$$

We introduce the subsets $\mathbb{B}$ and Ω^σ of $\mathbb{R}^4$ *via*

$$\mathbb{B} := \cup_{\zeta\in\mathbb{R}} \{\mathcal{B}_\zeta\times\{\zeta\}\}, \tag{4}$$

and

$$\Omega^\sigma := \mathbb{B}' = \cup_{\zeta\in\mathbb{R}} \{\mathcal{B}'_\zeta\times\{\zeta\}\}. \tag{5}$$

Whenever $A \subset \mathbb{R}^4$ and $\zeta \in \mathbb{R}$, the *ζ-section of* A, $A_\zeta \subset \mathbb{R}^3$, is defined by

$$A_\zeta := \{Y \in \mathbb{R}^3 \mid (Y,\zeta) \in A\}. \tag{6}$$

Evidently, we can write $A = \cup_{\zeta\in\mathbb{R}} \{A_\zeta\times\{\zeta\}\}$, while for the sets $\mathbb{B}$ and Ω^σ associated with the motion $\mathcal{M}$, we have $\mathbb{B}_\zeta = \mathcal{B}_\zeta$ and

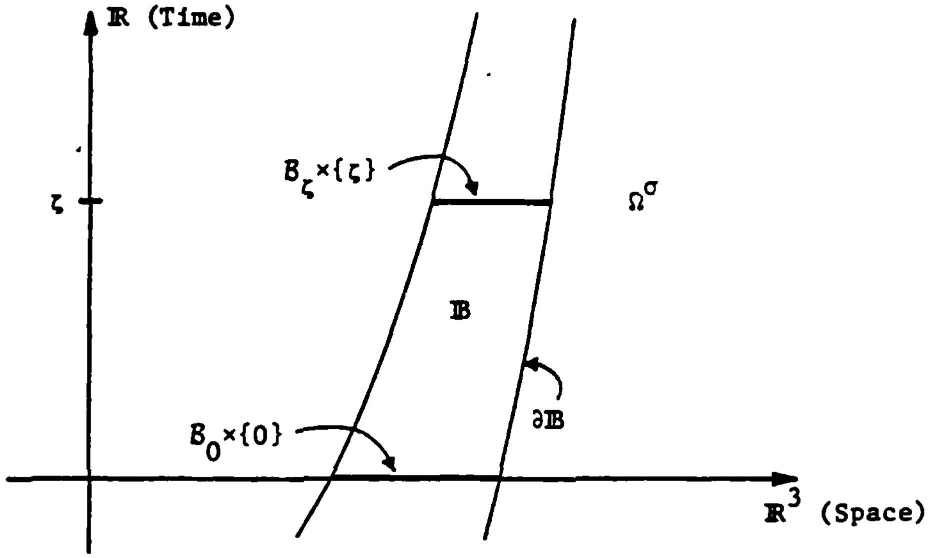

FIGURE 4. Body moving and deforming in space-time-alternate visualization

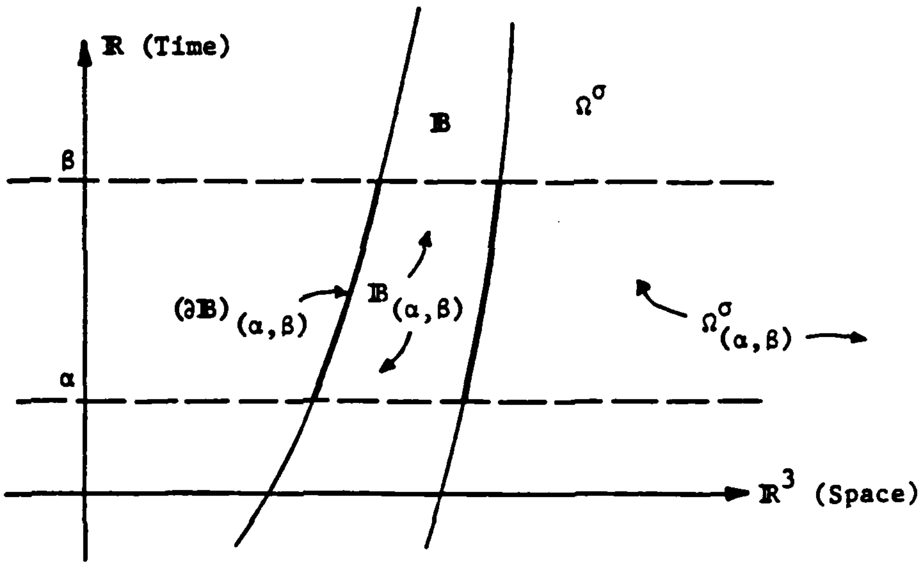

FIGURE 5. Sets associated with moving body

$\Omega^{\sigma}_{\zeta} = \mathcal{B}'_{\zeta}$ for each $\zeta \in \mathbb{R}$. Finally, if $A \subset \mathbb{R}^4$ and I is an interval in $\mathbb{R}$, we define $A_I \subset \mathbb{R}^4$ according to

$$A_I := \{(Y,\zeta) \in A \mid \zeta \in I\} = A \cap (\mathbb{R}^3 \times I). \tag{7}$$

[V.4] D E F I N I T I O N. A motion M is said to be *null*, or *stationary*, iff $c^* = 0$. ■.

[V.5] R E M A R K S. (a) Let M be a null motion. From (V.1.1), with $c^* = 0$, it is clear that $\chi(P,\zeta_1) = \chi(P,\zeta_2)$ if $P \in \partial\mathcal{R}$ and $\zeta_1, \zeta_2 \in \mathbb{R}$. Then $\partial\mathcal{B}_{\zeta_1} = \chi(\partial\mathcal{R},\zeta_1) = \chi(\partial\mathcal{R},\zeta_2) = \partial\mathcal{B}_{\zeta_2}$, with which reasoning as in [V.2.h.ii] shows that $\mathcal{B}_{\zeta_1} = \mathcal{B}_{\zeta_2}$ whenever $\zeta_1, \zeta_2 \in \mathbb{R}$ the body is stationary. Note, however, that it is easy to find an example of a nonstationary motion for which $\mathcal{B}_{\zeta_1} = \mathcal{B}_{\zeta_2}$ whenever $\zeta_1, \zeta_2 \in \mathbb{R}$; as a particularly simple one, consider a closed ball spinning about its fixed center. It turns out that if such a motion models the movement of a perfectly conducting body, the motion is "essentially null" insofar as its electromagnetic scattering properties are concerned, *i.e.*, the solution of a scattering problem associated with such a motion will be the same as the solution of the problem obtained by replacing the "essentially null" motion with the corresponding null motion, leaving all else the same.

(b) Obviously, with any set $\mathcal{B} \subset \mathbb{R}^3$ having all of the properties cited in [V.1.i], there is associated a unique null motion M: generate an element of $\mathbb{M}$ as $(\{\mathcal{B}_\zeta\}_{\zeta\in\mathbb{R}}, \mathcal{B}, \chi)$, wherein $\mathcal{B}_\zeta := \mathcal{B}$ for each $\zeta \in \mathbb{R}$, and $\chi(P,\zeta) := P$ for each $P \in \partial\mathcal{B}$ and $\zeta \in \mathbb{R}$. Take

M to be the $\mathcal{E}$-equivalence class in $\mathbb{M}$ which contains $(\{\mathcal{B}_\zeta\}_{\zeta\in\mathbb{R}}, \mathcal{B}, \chi)$.

(c) If we construct a triple $(\{\mathcal{B}_\zeta\}_{\zeta\in\mathbb{R}}, R, \chi)$ as in [V.1.i and ii] for which $c^* = 0$, then, just as in (a), we find that $\mathcal{B}_{\zeta_1} = \mathcal{B}_{\zeta_2}$ whenever $\zeta_1,\zeta_2 \in \mathbb{R}$. Then condition [V.1.iii] is certainly fulfilled. But if $\zeta \in \mathbb{R}$, $Z \in \partial\mathcal{B}_\zeta$, and $\varepsilon > 0$, we need only choose $Y_{I\varepsilon} \in B^3_\varepsilon(Z)\cap\mathcal{B}^o_\zeta$, $Y_{E\varepsilon} \in B^3_\varepsilon(Z)\cap\mathcal{B}'_\zeta$ and set $p_{I\varepsilon}(\xi) := Y_{I\varepsilon}$, $p_{E\varepsilon}(\xi) := Y_{E\varepsilon}$ for $|\xi| \leq \varepsilon$ to conclude that [V.1.iv] obtains, as well.

We continue to accumulate consequences of Definition [V.1]. For the proof of [V.7], we shall invoke the following general fact:

[V.6] L E M M A. *Let* T_1 *and* T_2 *be metric spaces. Suppose that* $f\colon T_1 \to T_2$ *is a continuous bijection such that whenever* K_2 *is a compact subset of* T_2, *there exists a corresponding compact* $K_1 \subset T_1$ *such that* $K_2 \subset f(K_1)$. *Then* f^{-1} *is continuous, i.e.,* f *is a homeomorphism.*

P R O O F. This follows from the somewhat more general result of Appendix V.A. □.

[V.7] P R O P O S I T I O N. *Let* M *be a motion,* (R,χ) *a reference pair for* M, *and* $\chi^*\colon \partial R\times\mathbb{R} \to \mathbb{R}^4$ *as in* [V.3]. χ^* *is a homeomorphism of* $\partial R\times\mathbb{R}$ *onto the set* $\cup_{\zeta\in\mathbb{R}} \{\partial\mathcal{B}_\zeta\times\{\zeta\}\}$.

P R O O F. We have $\chi^*(P,\zeta) := (\chi(P,\zeta),\zeta)$ for each $P \in \partial R$, $\zeta \in \mathbb{R}$, showing that the coordinate functions of χ^* are continuous on $\partial R\times\mathbb{R}$, since those of χ are. The continuity of χ^* follows.

For each $\zeta \in \mathbb{R}$, $\chi(\cdot,\zeta)$ is bijective from ∂R onto ∂B_ζ, so $\chi^*(\cdot,\zeta)$ is bijective from ∂R onto $\partial B_\zeta \times \{\zeta\}$; the bijectiveness of χ^* from $\partial R \times \mathbb{R}$ to $\cup_{\zeta\in\mathbb{R}} \{\partial B_\zeta \times \{\zeta\}\}$ follows readily. The continuity of χ^* as a map into $\mathbb{R}^4$ implies its continuity as a map onto $\cup_{\zeta\in\mathbb{R}} \{\partial B_\zeta \times \{\zeta\}\}$ equipped with the relative topology. To show that χ^{*-1}: $\cup_{\zeta\in\mathbb{R}} \{\partial B_\zeta \times \{\zeta\}\} \to \partial R \times \mathbb{R}$ is continuous, we apply Lemma [V.6]. Suppose, then, that $K \subset \cup_{\zeta\in\mathbb{R}} \{\partial B_\zeta \times \{\zeta\}\}$ is compact. Then K is bounded, so $|x^4| \leq N$ for each $x \in K$, for some $N > 0$. This gives $K \subset \cup_{\zeta\in[-N,N]} \{\partial B_\zeta \times \{\zeta\}\} = \chi^*(\partial R \times [-N,N])$. Noting that $\partial R \times [-N,N]$ is compact in $\partial R \times \mathbb{R}$, we see that χ^* has the property required of f in [V.6], when $T_1 = \partial R \times \mathbb{R}$ and $T_2 = \cup_{\zeta\in\mathbb{R}} \{\partial B_\zeta \times \{\zeta\}\}$ (which are, of course, equipped with the metric inherited from $\mathbb{R}^4$). Thus, χ^{*-1} is continuous. $\square$.

Observe that $\chi^{*-1}(Z,\zeta) = (\chi_\zeta^{-1}(Z),\zeta)$, for $\zeta \in \mathbb{R}$ and $Z \in \partial B_\zeta$. The coordinate functions of χ^{*-1} are continuous, by [V.7], so the coordinate functions of the map $(Z,\zeta) \mapsto \chi_\zeta^{-1}(Z)$ are also continuous. Consequently, we have proven

[V.8] C O R O L L A R Y. *Let M be a motion, and (R,χ) a reference pair for M. Then the map $(Z,\zeta) \mapsto \chi_\zeta^{-1}(Z)$ is continuous on $\cup_{\zeta\in\mathbb{R}} \{\partial B_\zeta \times \{\zeta\}\}$ onto ∂R.*

Statements [V.9] through [V.13] are expressions of the continuity, Lipschitz continuity, and "speed" properties of a motion. We shall rely heavily on these results in later arguments.

[V.9] L E M M A. *Let M be a motion. Suppose that $X \in \mathbb{R}^3$, with*

$X \in B^{o}_{\zeta_1} \cap B'_{\zeta_2}$ for some $\zeta_1, \zeta_2 \in \mathbb{R}$ (so $\zeta_1 \neq \zeta_2$). Then there exists at least one ζ between ζ_1 and ζ_2 for which $X \in \partial B_{\zeta}$.

P R O O F. Let I denote the closed interval with endpoints ζ_1 and ζ_2. Set $A_1 := \{\mu \in I | \ X \in B^{o}_{\mu}\}$ and $A_2 := \{\mu \in I | \ X \in B'_{\mu}\}$. Neither A_1 nor A_2 is empty, since $\zeta_1 \in A_1$ and $\zeta_2 \in A_2$. We shall show presently that A_1 and A_2 are mutually separated (*i.e.*, $A_1^{-} \cap A_2 = \emptyset$ and $A_1 \cap A_2^{-} = \emptyset$); if we assume, for the moment, that this has been accomplished, the proof of the lemma follows quickly. In fact, I is connected, so the union $A_1 \cup A_2$ of the non-void and mutually separated subsets A_1 and A_2 of I must be a proper subset of I. That is, there must exist some $\zeta \in I \cap (A_1 \cup A_2)'$, for which we then have $X \in B^{o\,\prime}_{\zeta} \cap B_{\zeta} = B'^{-}_{\zeta} \cap B_{\zeta} = \partial B_{\zeta}$, as required (obviously, $\zeta \neq \zeta_1$, $\zeta \neq \zeta_2$).

To show that A_1 and A_2 are mutually separated, we introduce the auxiliary function ψ: $I \rightarrow [0,\infty)$ given by

$$\psi(\mu) := \text{dist}\,(X, \partial B_{\mu}) := \inf\,\{|X-Y|_3| \ \ Y \in \partial B_{\mu}\} \quad \text{for each} \quad \mu \in I.$$

Selecting a reference pair (R, χ) for M, it is clear that we can also write

$$\psi(\mu) = \inf\,\{|X-\chi(P,\mu)|_3| \ \ P \in \partial R\} \quad \text{for each} \quad \mu \in I, \tag{1}$$

since $\chi(\partial R, \mu) = \partial B_{\mu}$, for each $\mu \in \mathbb{R}$. From (1), the continuity of ψ can be proven: letting $\varepsilon > 0$, (1) shows that we can find for each $\mu \in I$ a point $P_{\mu,\varepsilon} \in \partial R$ with $|X-\chi(P_{\mu,\varepsilon},\mu)|_3 < \psi(\mu)+(\varepsilon/2)$.

Then, if μ and ξ are in I, we have, with (1) and (V.1.1),

$$\psi(\xi)-\psi(\mu) < |X-\chi(P_{\mu,\varepsilon},\xi)|_3-|X-\chi(P_{\mu,\varepsilon},\mu)|_3+\frac{\varepsilon}{2}$$

$$\leq |\chi(P_{\mu,\varepsilon},\xi)-\chi(P_{\mu,\varepsilon},\mu)|_3+\frac{\varepsilon}{2}$$

$$\leq c^*|\xi-\mu|+\frac{\varepsilon}{2}.$$

Upon interchanging the roles of ξ and μ in this estimate and supposing that $|\mu-\xi| < \varepsilon/2c^*$, it follows that $|\psi(\xi)-\psi(\mu)| < \varepsilon$. Thus, ψ is continuous. To prove that $\bar{A}_1\cap A_2 = \emptyset$, assume, to the contrary, that there exists some $\mu_0 \in \bar{A}_1\cap A_2$. Then $X \in \mathcal{B}'_{\mu_0}$, so $\psi(\mu_0) > 0$. Also, we can find a sequence $(\mu_n)_{n=1}^{\infty}$ in A_1 which converges to μ_0; the continuity of ψ showing then that $\lim_{n\to\infty} \psi(\mu_n) = \psi(\mu_0)$, we shall suppose that $\psi(\mu_n) > \frac{1}{2}\psi(\mu_0)$ for each $n \in \mathbb{N}$, as we may without loss. Setting $\delta := \frac{1}{2}\ \psi(\mu_0)$, it is clear that $B^3_\delta(X) \subset \mathcal{B}_{\mu_n}\cap\mathcal{B}'_{\mu_0}$, since $\{\mu_n\}_{n=1}^{\infty} \subset A_1$ and $\mu_0 \in A_2$, so

$$\lambda_3(\mathcal{B}_{\mu_n}\cap\mathcal{B}'_{\mu_0}) \geq \frac{4}{3}\pi\delta^3 > 0, \quad \text{for each} \quad n \in \mathbb{N}.$$

However, this contradicts property [V.1.iv], which says that

$$\lambda_3(\mathcal{B}_{\mu_n}\cap\mathcal{B}'_{\mu_0}) \leq \lambda_3(\{\mathcal{B}_{\mu_n}\cap\mathcal{B}'_{\mu_0}\}\cup\{\mathcal{B}'_{\mu_n}\cap\mathcal{B}_{\mu_0}\})$$

$$= \lambda_3(\mathcal{B}_{\mu_n}\Delta\mathcal{B}_{\mu_0}) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $\bar{A}_1\cap A_2 = \emptyset$. Similarly, if we suppose that $A_1\cap\bar{A}_2$ contains a point ξ_0, we can find a sequence $(\xi_n)_{n=1}^{\infty}$ in A_2 and converging to ξ_0 for which

$$\lambda_3(\mathcal{B}'_{\xi_n} \cap \mathcal{B}_{\xi_0}) \geq \frac{4}{3}\pi\tilde{\delta}^3 \qquad \text{for each} \quad n \in \mathbb{N},$$

wherein $\tilde{\delta} := \frac{1}{2}\psi(\xi_0) > 0$. This is again impossible by [V.1.iv], for we must have

$$\lambda_3(\mathcal{B}'_{\xi_n} \cap \mathcal{B}_{\xi_0}) \leq \lambda_3(\mathcal{B}_{\xi_n} \Delta \mathcal{B}_{\xi_0}) \to 0 \qquad \text{as} \qquad n \to \infty.$$

Thus, A_1 and A_2 are mutually separated. As noted, this completes the proof of the lemma. □.

[V.10] L E M M A. *Let* M *be a motion. If* $\zeta \in \mathbb{R}$ *and* $\tilde{c} > c^*$, *then* $\mathcal{B}_{\zeta+\xi} \subset B^3_{\tilde{c}|\xi|}(\mathcal{B}_\zeta)$ *for each* $\xi \in \mathbb{R}$, $\xi \neq 0$.

P R O O F. Choose $\xi \in \mathbb{R}$, $\xi \neq 0$. Let $Y \in B^3_{\tilde{c}|\xi|}(\mathcal{B}_\zeta)'$, so that dist $(Y, \partial\mathcal{B}_\zeta) := \inf\{|Z-Y|_3 \mid Z \in \partial\mathcal{B}_\zeta\} \geq \tilde{c}|\xi|$, and $Y \in \mathcal{B}'_\zeta$. Suppose, however, that $Y \in \mathcal{B}_{\zeta+\xi} = \mathcal{B}^o_{\zeta+\xi} \cup \partial\mathcal{B}_{\zeta+\xi}$. From Lemma [V.9], we conclude that $Y \in \partial\mathcal{B}_\mu$ for some $\mu \in I$, where I is the half-open interval with endpoints ζ and $\zeta+\xi$, including $\zeta+\xi$. Choose a reference pair $(\mathcal{B}_\mu, \chi^\mu)$ for M, where $\chi^\mu(\cdot,\mu)$ is the identity on $\partial\mathcal{B}_\mu$; cf., [V.2.j]. Then, on the one hand, the condition dist $(Y, \partial\mathcal{B}_\zeta) \geq \tilde{c}|\xi|$ gives, since $\chi^\mu(Y,\zeta) \in \partial\mathcal{B}_\zeta$,

$$|\chi^\mu(Y,\zeta+\xi)-Y|_3 + |\chi^\mu(Y,\zeta)-Y|_3 \geq \tilde{c}|\xi|,$$

while, on the other,

$$|\chi^\mu(Y,\zeta+\xi)-Y|_3 + |\chi^\mu(Y,\zeta)-Y|_3 = |\chi^\mu(Y,\zeta+\xi)-\chi^\mu(Y,\mu)|_3 + |\chi^\mu(Y,\zeta)-\chi^\mu(Y,\mu)|_3$$

$$\leq c^*\{|\zeta+\xi-\mu| + |\zeta-\mu|\} = c^*|\xi| < \tilde{c}|\xi|.$$

This contradiction implies that $Y \in \mathcal{B}'_{\zeta+\xi}$, so $B^3_{\tilde{c}|\xi|}(\mathcal{B}_\zeta)' \subset \mathcal{B}'_{\zeta+\xi}$, which gives the desired conclusion. $\square$.

For future convenience, the following immediate consequences of [V.10] are set down:

[V.11] C O R O L L A R Y. *Let* M *be a motion,* $\zeta \in \mathbb{R}$, *and* $\tilde{c} \in (c^*, c)$.

(*i*) $\mathcal{B}_{\zeta-\xi} \subset B^3_{\tilde{c}\xi}(\mathcal{B}_\zeta)$ *for each* $\xi > 0$.

(*ii*) *If* $\varepsilon > 0$, *then* $\mathcal{B}_\xi \subset B^3_\varepsilon(\mathcal{B}_\zeta)$ *whenever* $\xi \in \mathbb{R}$, $|\xi-\zeta| \leq \varepsilon/\tilde{c}$.

[V.12] L E M M A. *Let* M *be a nonstationary motion (so* $c^* > 0$*). Suppose* $Z \in \mathbb{R}^3$, $\zeta \in \mathbb{R}$.

(*i*) *If* $Z \in \mathcal{B}^o_\zeta$, *then* $Z \in \mathcal{B}^o_{\zeta+\xi}$ *for* $|\xi| < \frac{1}{c^*}\,\mathrm{dist}\,(Z, \partial\mathcal{B}_\zeta)$.

(*ii*) *If* $Z \in \mathcal{B}'_\zeta$, *then* $Z \in \mathcal{B}'_{\zeta+\xi}$ *for* $|\xi| < \frac{1}{c^*}\,\mathrm{dist}\,(Z, \partial\mathcal{B}_\zeta)$.

(*iii*) *If* $Z \in \partial\mathcal{B}_\zeta$ *and* $\delta > 0$, *then* $B^3_\delta(Z) \cap \mathcal{B}^o_{\zeta+\xi} \neq \emptyset$ *and* $B^3_\delta(Z) \cap \mathcal{B}'_{\zeta+\xi} \neq \emptyset$ *for* $|\xi| < \frac{1}{c^*}\,\delta$.

P R O O F. (i) Set $\eta := \mathrm{dist}\,(Z, \partial\mathcal{B}_\zeta)$; then $\eta > 0$. Fix $\xi \in \mathbb{R}$, with $0 < |\xi| < \frac{1}{c^*}\eta$, and suppose, contrary to the conclusion, that $Z \in \mathcal{B}^{o\,\prime}_{\zeta+\xi} = \mathcal{B}'^{-}_{\zeta+\xi} = \mathcal{B}'_{\zeta+\xi} \cup \partial\mathcal{B}_{\zeta+\xi}$. Let I denote the half-open interval including $\zeta+\xi$, with endpoints ζ and $\zeta+\xi$. Using Lemma [V.9], there must exist $\mu \in I$ such that $Z \in \partial\mathcal{B}_\mu$, whence $Z = \chi(P_Z, \mu)$ for some $P_Z \in \partial R$; (R, χ) is any reference pair for M. Since $Z \in \mathcal{B}^o_\zeta$,

$\chi(P_Z,\zeta) \in \partial\mathcal{B}_\zeta$, we must have

$$|\chi(P_Z,\zeta)-\chi(P_Z,\mu)|_3+|\chi(P_Z,\zeta+\xi)-\chi(P_Z,\mu)|_3 \geq \eta,$$

while the sum on the left in this inequality is also $\leq c^*\{|\zeta-\mu|+|\zeta+\xi-\mu|\} = c^*|\xi| < \eta$. This impossibility shows that $Z \in \mathcal{B}^o_{\zeta+\xi}$, and completes the proof of (i).

(ii) Retrace the steps of the proof of (i), *mutatis mutandis*.

(iii) Choose any reference pair $(\mathcal{R},\chi)$ for $\mathcal{M}$. Then $Z = \chi(P_Z,\zeta)$ for some $P_Z \in \partial\mathcal{R}$. Fix $\xi \in \mathbb{R}$ with $|\xi| < \frac{1}{c^*}\delta$. Since

$$|\chi(P_Z,\zeta+\xi)-\chi(P_Z,\zeta)|_3 \leq c^*|\xi| < \delta,$$

we conclude that $\chi(P_Z,\zeta+\xi) \in B^3_\delta(Z)$, *i.e.*, $B^3_\delta(Z)$ is a neighborhood of $\chi(P_Z,\zeta+\xi) \in \partial\mathcal{B}_{\zeta+\xi}$. Obviously, then, $B^3_\delta(Z)\cap\mathcal{B}'_{\zeta+\xi} \neq \emptyset$. But also, $\mathcal{B}_{\zeta+\xi}$ is regularly closed, *i.e.*, $\mathcal{B}_{\zeta+\xi} = \mathcal{B}^{o-}_{\zeta+\xi}$; since $\chi(P_Z,\zeta+\xi) \in \mathcal{B}_{\zeta+\xi}$, $\chi(P_Z,\zeta+\xi)$ is the limit of a sequence of points in $\mathcal{B}^o_{\zeta+\xi}$, so the neighborhood $B^3_\delta(Z)$ of $\chi(P_Z,\zeta+\xi)$ must meet $\mathcal{B}^o_{\zeta+\xi}$, as well. This completes the proof. $\square$.

Lemma [V.12] clearly implies that, for any motion $\mathcal{M}$ and any $Z \in \mathbb{R}^3$, the sets $\{\zeta \in \mathbb{R}|\ \ Z \in \mathcal{B}^o_\zeta\}$, $\{\zeta \in \mathbb{R}|\ \ Z \in \mathcal{B}'_\zeta\}$ are each open in $\mathbb{R}$ (so that $\{\zeta \in \mathbb{R}|\ \ Z \in \partial\mathcal{B}_\zeta\}$ is closed, a fact which can also be proven more directly).

[V.13] L E M M A. *Let* $\mathcal{M}$ *be a motion,* $Z \in \mathbb{R}^3$, $\zeta \in \mathbb{R}$, *and* $\varepsilon > 0$.

(*i*) *If* $B^3_\varepsilon(Z) \subset \mathcal{B}_\zeta$, *then* $B^3_{\varepsilon-c|\xi-\zeta|}(Z)^- \subset \mathcal{B}^o_\xi$ *whenever*

$\xi \in \mathbb{R}$ *with* $|\xi-\zeta| \leq \varepsilon/c$;

(*ii*) *If* $B^3_\varepsilon(Z) \subset B'^-_\zeta$, *then* $B^3_{\varepsilon-c|\xi-\zeta|}(Z)^- \subset B'_\xi$ *whenever* $\xi \in \mathbb{R}$ *with* $|\xi-\zeta| \leq \varepsilon/c$;

recall the convention $B^3_0(Z)^- := \{Z\}$.

P R O O F. (i) Observe that we actually have $B^3_\varepsilon(Z) \subset \mathcal{B}^o_\zeta$. Suppose that $\xi \in \mathbb{R}$ and $0 < |\xi-\zeta| \leq \varepsilon/c$ (if $\xi = \zeta$, the desired conclusion has already been pointed out); assume that $Y \in \mathbb{R}^3$ with $|Y-Z|_3 \leq \varepsilon-c|\xi-\zeta|$ $(< \varepsilon)$, but $Y \in \mathcal{B}^{o\,\prime}_\xi = B'^-_\xi = B'_\xi \cup \partial\mathcal{B}_\xi$. Since $Y \in B^3_\varepsilon(Z) \subset \mathcal{B}^o_\zeta$, [V.9] shows that there is some $\mu \in \mathbb{R}$, either lying between ξ and ζ or equal to ξ, such that $Y \in \partial\mathcal{B}_\mu$. For convenience, choose the reference pair $(\mathcal{B}_\mu, \chi^\mu)$ for M, where $\chi^\mu(\cdot,\mu)$ is the identity on $\partial\mathcal{B}_\mu$. Then

$$\begin{aligned}|\chi^\mu(Y,\zeta)-Y|_3+|\chi^\mu(Y,\xi)-Y|_3 &\geq |\chi^\mu(Y,\zeta)-Y|_3\\ &\geq |\chi^\mu(Y,\zeta)-Z|_3-|Y-Z|_3\\ &\geq \varepsilon-(\varepsilon-c|\xi-\zeta|)\\ &= c|\xi-\zeta| > c^*|\xi-\zeta|\end{aligned}$$

(having noted that $\chi^\mu(Y,\zeta) \in \partial\mathcal{B}_\zeta$, so $|\chi^\mu(Y,\zeta)-Z|_3 \geq \varepsilon$), but also

$$\begin{aligned}|\chi^\mu(Y,\zeta)-Y|_3+|\chi^\mu(Y,\xi)-Y|_3 &= |\chi^\mu(Y,\zeta)-\chi^\mu(Y,\mu)|_3+|\chi^\mu(Y,\xi)-\chi^\mu(Y,\mu)|_3\\ &\leq c^*(|\zeta-\mu|+|\xi-\mu|) = c^*|\xi-\zeta|;\end{aligned}$$

this impossibility implies that $Y \in \mathcal{B}^o_\xi$. Thus, $B^3_{\varepsilon-c|\xi-\zeta|}(Z)^- \subset \mathcal{B}^o_\xi$, and the proof of (i) is complete.

(ii) Observe that we actually have $B^3_\varepsilon(Z) \subset B'^{-o}_\zeta$; since B'_ζ is regularly open, it follows that $B^3_\varepsilon(Z) \subset B'_\zeta$. Assume that $Y \in \mathbb{R}^3$ with $|Y-Z|_3 \leq \varepsilon - c|\xi-\zeta|$, but $Y \in B_\xi$, where $\xi \in \mathbb{R}$ has been chosen with $0 < |\xi-\zeta| \leq \varepsilon/c$. Since $Y \in B^3_\varepsilon(Z) \subset B'_\zeta$, [V.9] implies that there exists a $\mu \in \mathbb{R}$ with the properties listed in the proof of (i). Retrace the steps of the proof of (i) to arrive at a contradiction, forcing the conclusion that $Y \in B'_\xi$, hence that $B^3_{\varepsilon-c|\xi-\zeta|}(Z)^- \subset B'_\xi$. □.

Recall the definition of the set $A_\zeta \subset \mathbb{R}^3$ associated with any $A \subset \mathbb{R}^4$, $\zeta \in \mathbb{R}$; cf., [V.3].

[V.14] L E M M A. *Let* $A \subset \mathbb{R}^4$ *and* $\zeta \in \mathbb{R}$. *Then*

(i) $(A^o)_\zeta \subset (A_\zeta)^o$;

(ii) $(A_\zeta)^- \subset (A^-)_\zeta$;

(iii) $\partial(A_\zeta) \subset (\partial A)_\zeta$.

P R O O F. (i) Suppose $Y \in (A^o)_\zeta$. Then $(Y,\zeta) \in A^o$, so there is some $\varepsilon > 0$ such that $\{r^2_Y(Z)+|\xi-\zeta|^2\}^{1/2} < \varepsilon$ implies $(Z,\xi) \in A$. Thus, in particular, $r_Y(Z) < \varepsilon$ implies $(Z,\zeta) \in A$, or $Z \in A_\zeta$. Consequently, $B^3_\varepsilon(Y) \subset A_\zeta$, so $Y \in (A_\zeta)^o$.

(ii) Suppose $Y \in (A_\zeta)^-$. Then $B^3_\varepsilon(Y)\cap A_\zeta \neq \emptyset$ for each $\varepsilon > 0$, so for each $\varepsilon > 0$ there is some $Y_\varepsilon \in \mathbb{R}^3$ such that $r_Y(Y_\varepsilon) < \varepsilon$ and $Y_\varepsilon \in A_\zeta$. Then, for each $\varepsilon > 0$, $(Y_\varepsilon,\zeta) \in B^4_\varepsilon(Y,\zeta)\cap A$, giving $(Y,\zeta) \in A^-$, so that $Y \in (A^-)_\zeta$.

(iii)

$$\partial(A_\zeta) = (A_\zeta)^- \cap (A_\zeta)'^- = (A_\zeta)^- \cap ((A')_\zeta)^- \subset (A^-)_\zeta \cap (A'^-)_\zeta$$
$$= (A^- \cap A'^-)_\zeta = (\partial A)_\zeta,$$

where we have used (ii) and the obvious facts that $(A_\zeta)' = (A')_\zeta$, and $A_{1\zeta} \cap A_{2\zeta} = (A_1 \cap A_2)_\zeta$, for $A_1, A_2 \subset \mathbb{R}^4$. □.

We are now prepared to prove that the sets $\mathbb{B}$ and Ω^σ, intrinsic to a motion M, possess a number of desired properties.

[V.15] T H E O R E M. *Let* M *be a motion, and* $\mathbb{B} \subset \mathbb{R}^4$, $\Omega^\sigma \subset \mathbb{R}^4$ *the associated sets as in* [V.3]. *Then*

(i) $\mathbb{B}$ *is closed;* Ω^σ *is open;*

(ii) $\mathbb{B}^\circ = \cup_{\zeta \in \mathbb{R}} \{B_\zeta^\circ \times \{\zeta\}\}$; $\Omega^{\sigma -} = \cup_{\zeta \in \mathbb{R}} \{B_\zeta'^- \times \{\zeta\}\}$;

(iii) $\partial\mathbb{B} = \partial\Omega^\sigma = \cup_{\zeta \in \mathbb{R}} \{\partial B_\zeta \times \{\zeta\}\}$;

(iv) $\mathbb{B}$ *is regularly closed;* Ω^σ *is regularly open;*

(v) Ω^σ *is connected;*

(vi) if (R,χ) *is a reference pair for* M, *then* χ^* *provides a homeomorphism of* $\partial R \times \mathbb{R}$ *onto* $\partial\mathbb{B}$.

P R O O F. (i) Let $((Y_n,\zeta_n))_1^\infty \subset \mathbb{B} := \cup_{\zeta \in \mathbb{R}} \{B_\zeta \times \{\zeta\}\}$ be a sequence converging to $(Y_0,\zeta_0) \in \mathbb{R}^4$. Note that $Y_n \in B_{\zeta_n}$ for each $n \in \mathbb{N}$, $\lim_{n \to \infty} Y_n = Y_0$, and $\lim_{n \to \infty} \zeta_n = \zeta_0$. The proof that $\mathbb{B}$ is closed is complete once it has been shown that $(Y_0,\zeta_0) \in \mathbb{B}$, *i.e.*, that

$Y_0 \in \mathcal{B}_{\zeta_0}$. Now, if $\varepsilon > 0$, there is an $n(\varepsilon) \in \mathbb{N}$ such that $r_{Y_0}(Y_n) < \varepsilon$ and $|\zeta_n - \zeta_0| < \varepsilon$ whenever $n \geq n(\varepsilon)$. According to [V.11.ii], we then have $\mathcal{B}_{\zeta_n} \subset B^3_\varepsilon(\mathcal{B}_{\zeta_0})$ whenever $n \geq n(\varepsilon/\tilde{c})$, where $\tilde{c} \in (c^*, c)$. Assume that $Y_0 \in \mathcal{B}'_{\zeta_0}$: since $\mathcal{B}_{\zeta_0}$ is closed, $\Delta := \text{dist}\,(Y, \mathcal{B}_{\zeta_0}) > 0$, and it is easy to see that $B^3_{\Delta/3}(Y_0) \cap B^3_{\Delta/3}(\mathcal{B}_{\zeta_0}) = \emptyset$. However, choosing $n \geq \max\{n(\Delta/3), n(\Delta/3\tilde{c})\}$, we find both $Y_n \in B^3_{\Delta/3}(Y_0)$ and $Y_n \in \mathcal{B}_{\zeta_n} \subset B^3_{\Delta/3}(\mathcal{B}_{\zeta_0})$. This contradiction implies that $Y_0 \in \mathcal{B}_{\zeta_0}$, as required. Thus, $\mathbb{B}$ is closed. Then $\Omega^\sigma := \mathbb{B}'$ is open.

(ii) Recall that $\mathbb{B}_\zeta = \mathcal{B}_\zeta$ for each $\zeta \in \mathbb{R}$. Suppose we have shown that $\mathcal{B}^o_\zeta \subset (\mathbb{B}^o)_\zeta$ for each $\zeta \in \mathbb{R}$; then, [V.14.i] giving the reversed inclusion $(\mathbb{B}^o)_\zeta \subset (\mathbb{B}_\zeta)^o = \mathcal{B}^o_\zeta$, we shall have $(\mathbb{B}^o)_\zeta = \mathcal{B}^o_\zeta$ for all ζ, whence $\mathbb{B}^o = \cup_{\zeta\in\mathbb{R}} \{(\mathbb{B}^o)_\zeta \times \{\zeta\}\} = \cup_{\zeta\in\mathbb{R}} \{\mathcal{B}^o_\zeta \times \{\zeta\}\}$, and $\Omega^{\sigma-} = \mathbb{B}'^{-} = \mathbb{B}^{o\prime} = \cup_{\zeta\in\mathbb{R}} \{\mathcal{B}^{o\prime}_\zeta \times \{\zeta\}\} = \cup_{\zeta\in\mathbb{R}} \{\mathcal{B}'^{-}_\zeta \times \{\zeta\}\}$. Thus, the entire statement of (ii) will follow by establishing the inclusion $\mathcal{B}^o_\zeta \subset (\mathbb{B}^o)_\zeta$ for each $\zeta \in \mathbb{R}$.

Accordingly, fix $\zeta \in \mathbb{R}$, and suppose $Y \in \mathcal{B}^o_\zeta$. Then $B^3_\varepsilon(Y) \subset \mathcal{B}_\zeta$ for some $\varepsilon > 0$. By [V.13.i], whenever $\xi \in \mathbb{R}$ with $|\zeta - \xi| \leq \varepsilon/c$, *i.e.*, $\zeta - \frac{1}{c}\varepsilon \leq \xi \leq \zeta + \frac{1}{c}\varepsilon$, then $B^3_{\varepsilon - c|\zeta-\xi|}(Y) \subset \mathcal{B}^o_\xi$. With this fact, it is easy to find a neighborhood of (Y, ζ) lying in $\mathbb{B}$. Indeed, the open "pillbox" $B^3_{\varepsilon/2}(Y) \times \left(\zeta - \frac{\varepsilon}{2c}, \ \zeta + \frac{\varepsilon}{2c}\right)$ contains (Y, ζ) and is also in $\mathbb{B}$, for, whenever $\zeta - \frac{\varepsilon}{2c} < \xi < \zeta + \frac{\varepsilon}{2c}$, then $\varepsilon - c|\zeta - \xi| > \varepsilon/2$, so $B^3_{\varepsilon/2}(Y) \subset B^3_{\varepsilon - c|\zeta-\xi|}(Y) \subset \mathcal{B}_\xi$; thus,

$$B^3_{\varepsilon/2}(Y)\times\left(\zeta-\frac{\varepsilon}{2c}\ ,\ \zeta+\frac{\varepsilon}{2c}\right) = \cup_{\zeta-\frac{\varepsilon}{2c}<\xi<\zeta+\frac{\varepsilon}{2c}} \{B^3_{\varepsilon/2}(Y)\times\{\xi\}\}$$

$$\subset \cup_{\zeta-\frac{\varepsilon}{2c}<\xi<\zeta+\frac{\varepsilon}{2c}} \{\mathcal{B}_\xi\times\{\xi\}\} \subset \mathbb{B}.$$

Consequently, $(Y,\zeta)\in\mathbb{B}^o$, whence $Y\in(\mathbb{B}^o)_\zeta$. This proves that $\mathcal{B}^o_\zeta\subset(\mathbb{B}^o)_\zeta$ for each $\zeta\in\mathbb{R}$, from which statement (ii) follows, as we have already decided.

(iii) Choose $\zeta\in\mathbb{R}$. We shall show that $(\partial\mathbb{B})_\zeta\subset\partial(\mathbb{B}_\zeta)$: again, since $\mathbb{B}_\zeta=\mathcal{B}_\zeta$, we have $\partial(\mathbb{B}_\zeta)=\partial\mathcal{B}_\zeta$. Let $Y\in(\partial\mathbb{B})_\zeta$, so $(Y,\zeta)\in\partial\mathbb{B}$. We cannot have $Y\in\mathcal{B}^o_\zeta$, for then, by (ii), $Y\in(\mathbb{B}^o)_\zeta$, giving $(Y,\zeta)\in\mathbb{B}^o$, which contradicts $(Y,\zeta)\in\partial\mathbb{B}$. Thus, $Y\in\mathcal{B}^{o\prime}_\zeta$. Also, since $\mathbb{B}$ is closed, $(Y,\zeta)\in\partial\mathbb{B}$ implies $(Y,\zeta)\in\mathbb{B}$, so $Y\in\mathbb{B}_\zeta=\mathcal{B}_\zeta$. Hence, $Y\in\mathcal{B}_\zeta\cap\mathcal{B}^{o\prime}_\zeta=\mathcal{B}_\zeta\cap\mathcal{B}^{\prime-}_\zeta=\partial\mathcal{B}_\zeta$ ($\mathcal{B}_\zeta$ is closed). This shows that $(\partial\mathbb{B})_\zeta\subset\partial(\mathbb{B}_\zeta)=\partial\mathcal{B}_\zeta$. Meanwhile, [V.14.iii] gives $\partial(\mathbb{B}_\zeta)\subset(\partial\mathbb{B})_\zeta$. We have then shown that $(\partial\mathbb{B})_\zeta=\partial\mathcal{B}_\zeta$, for each $\zeta\in\mathbb{R}$, which implies the assertion of (iii), for now we can write

$$\partial\mathbb{B}=\cup_{\zeta\in\mathbb{R}}\{(\partial\mathbb{B})_\zeta\times\{\zeta\}\}=\cup_{\zeta\in\mathbb{R}}\{\partial\mathcal{B}_\zeta\times\{\zeta\}\}.$$

(iv) We must show that $\mathbb{B}=\mathbb{B}^{o-}$. Since $\mathbb{B}$ is closed, $\mathbb{B}^{o-}\subset\mathbb{B}$. Now let $(Y,\zeta)\in\mathbb{B}$, so $Y\in\mathcal{B}_\zeta=\mathcal{B}^{o-}_\zeta$, the latter equality holding since $\mathcal{B}_\zeta$ is regularly closed ([V.1.i]). Therefore, there is a sequence $(Y_i)_1^\infty$ in $\mathcal{B}^o_\zeta$ converging to Y.

Certainly, the sequence $((Y_i,\zeta))_1^\infty$ converges to (Y,ζ), and lies in $B_\zeta^o\times\{\zeta\}\subset \mathbb{B}^o$, by (ii). This shows that $(Y,\zeta)\in \mathbb{B}^{o-}$, and completes the proof of the fact that $\mathbb{B}$ is regularly closed. But then Ω^σ is regularly open, since $\Omega^\sigma = \mathbb{B}'$.

(v) For each $\zeta\in\mathbb{R}$, B_ζ' is open and connected in $\mathbb{R}^3$, so it is pathwise connected. Then $B_\zeta'\times\{\zeta\}$ is pathwise connected in $\mathbb{R}^4$. If we construct a continuous function $p\colon \mathbb{R}\to\mathbb{R}^3$ such that $p(\zeta)\in B_\zeta'$ for each $\zeta\in\mathbb{R}$, then the function $p^*\colon \mathbb{R}\to\mathbb{R}^4$ given by $p^*(\zeta) := (p(\zeta),\zeta)$, $\zeta\in\mathbb{R}$, shall be continuous, with $p^*(\zeta)\in B_\zeta'\times\{\zeta\}$ for each $\zeta\in\mathbb{R}$. The pathwise connectedness of $\cup_{\zeta\in\mathbb{R}}\{B_\zeta'\times\{\zeta\}\} = \Omega^\sigma$ would be an immediate consequence of these facts, whence the connectedness of Ω^σ would follow, in turn. Thus, the proof rests upon showing that there is a function p with the required properties. To construct one, choose $Y\in\mathbb{R}^3$ and $\rho>0$ such that $B_0\subset B_\rho^3(Y)$. Since $B_\zeta\subset B^3_{\tilde c|\zeta|}(B_0)$ for each $\zeta\in\mathbb{R}$, $\zeta\neq 0$, by [V.10], where $\tilde c$ has been chosen in (c^*,c), it follows easily that $B_\zeta\subset B^3_{\tilde\rho+\tilde c|\zeta|}(Y)$ for each $\zeta\in\mathbb{R}$, where $\tilde\rho := \rho+\operatorname{diam} B_0$ (in view of the inequality $r_Y(Z)\leq \operatorname{dist}(Y,B_0)+\operatorname{dist}(Z,B_0)+\operatorname{diam} B_0$, $Z\in\mathbb{R}^3$). Thus, we have merely to exhibit a continuous function $p\colon\mathbb{R}\to\mathbb{R}^3$ such that $p(\zeta)\in B^3_{\tilde\rho+\tilde c|\zeta|}(Y)'$ for each $\zeta\in\mathbb{R}$, so it suffices to choose any $\gamma\in\partial B_1^3(0)$ and set $p(\zeta) := Y+(\tilde\rho+\tilde c|\zeta|)\gamma$, $\zeta\in\mathbb{R}$.

(vi) In [V.7], it was demonstrated that χ^* provides a homeomorphism of $\partial R\times\mathbb{R}$ onto $\cup_{\zeta\in\mathbb{R}}\{\partial B_\zeta\times\{\zeta\}\}$, but now (iii) says that the latter set is just $\partial\mathbb{B}$. $\square$.

It is essential, for later developments, to examine the set-theoretic and topological connections between a motion (in particular, between the sets $\mathbb{B}$ and Ω^σ) and characteristic cones in $\mathbb{R}^4$. We now digress to prepare some general facts in this direction, later returning to consider their implications in the setting of present interest.

[V.16] DEFINITIONS. (i) Let $X\in\mathbb{R}^3$, $t\in\mathbb{R}$. The *backward* [*forward*] *characteristic*, or *light, cone with vertex at* (X,t) is the set $C_-(X,t)$ $[C_+(X,t)]\subset\mathbb{R}^4$ given by

$$C_{\underset{[+]}{-}}(X,t) := \{(Y,\zeta)\in\mathbb{R}^4 \mid Y\in\mathbb{R}^3,\ \zeta = t_{\underset{[+]}{-}}\frac{1}{c}\, r_X(Y)\}.$$

It is easy to show then that $C_-(X,t) = \cup_{\xi\le t}\{\partial B^3_{c(t-\xi)}(X)\times\{\xi\}\}$ and $C_+(X,t) = \cup_{\xi\ge t}\{\partial B^3_{c(\xi-t)}(X)\times\{\xi\}\}$, where $B^3_0(X) := \{X\}$.

(ii) Let $\mathrm{Pr}\colon \mathbb{R}^4\to\mathbb{R}^3$ denote the projection map $(Y,\zeta)\mapsto Y$, for $Y\in\mathbb{R}^3$, $\zeta\in\mathbb{R}$.

(iii) Let $X\in\mathbb{R}^3$, $t\in\mathbb{R}$. Define $P_{(X,t)}\colon \mathbb{R}^3\to\mathbb{R}^4$ by

$$P_{(X,t)}(Y) := (Y,\ t-\frac{1}{c}\, r_X(Y)) \qquad \text{for each} \qquad Y\in\mathbb{R}^3.$$

Then, for $A \subset \mathbb{R}^4$, define the *retardation of* A *relative to* (X,t) to be the set $A(X,t) \subset \mathbb{R}^3$ given by

$$A(X,t) := P^{-1}_{(X,t)}(A) = \{Y \in \mathbb{R}^3 \mid \; P_{(X,t)}(Y) = (Y,\, t - \tfrac{1}{c}\, r_X(Y)) \in A\}. \quad \square.$$

[V.17] R E M A R K S. (a) The continuity of Pr and $P_{(X,t)}$, for each $(X,t) \in \mathbb{R}^4$, is obvious.

(b) If $X \in \mathbb{R}^3$, $t \in \mathbb{R}$, $P_{(X,t)}$ is clearly just the inverse of the continuous bijection $Pr|\; C_-(X,t)\colon\; C_-(X,t) \to \mathbb{R}^3$, so $P_{(X,t)}$ is a homeomorphism of $\mathbb{R}^3$ onto $C_-(X,t)$.

(c) For $A \subset \mathbb{R}^4$, $\zeta \in \mathbb{R}$, the ζ-section of A, A_ζ (cf., [V.3]), is easily seen to be just $Pr\,(A \cap \{\mathbb{R}^3 \times \{\zeta\}\})$.

The following alternate characterizations of the retardation of a set with respect to a point are useful; the first and third also afford a convenient means for visualization; cf., Figures 6 and 7.

[V.18] P R O P O S I T I O N. *Let* $A \subset \mathbb{R}^4$, $X \in \mathbb{R}^3$, *and* $t \in \mathbb{R}$. *Then*

(*i*) $A(X,t) = Pr\,(A \cap C_-(X,t))$; *in particular,* $A(X,t)$ *is non-void iff* $A \cap C_-(X,t)$ *is non-void;*

(*ii*) $A(X,t) = \left\{ Y \in \mathbb{R}^3 \mid \; Y \in A_{t - \frac{1}{c} r_X(Y)} \right\};$

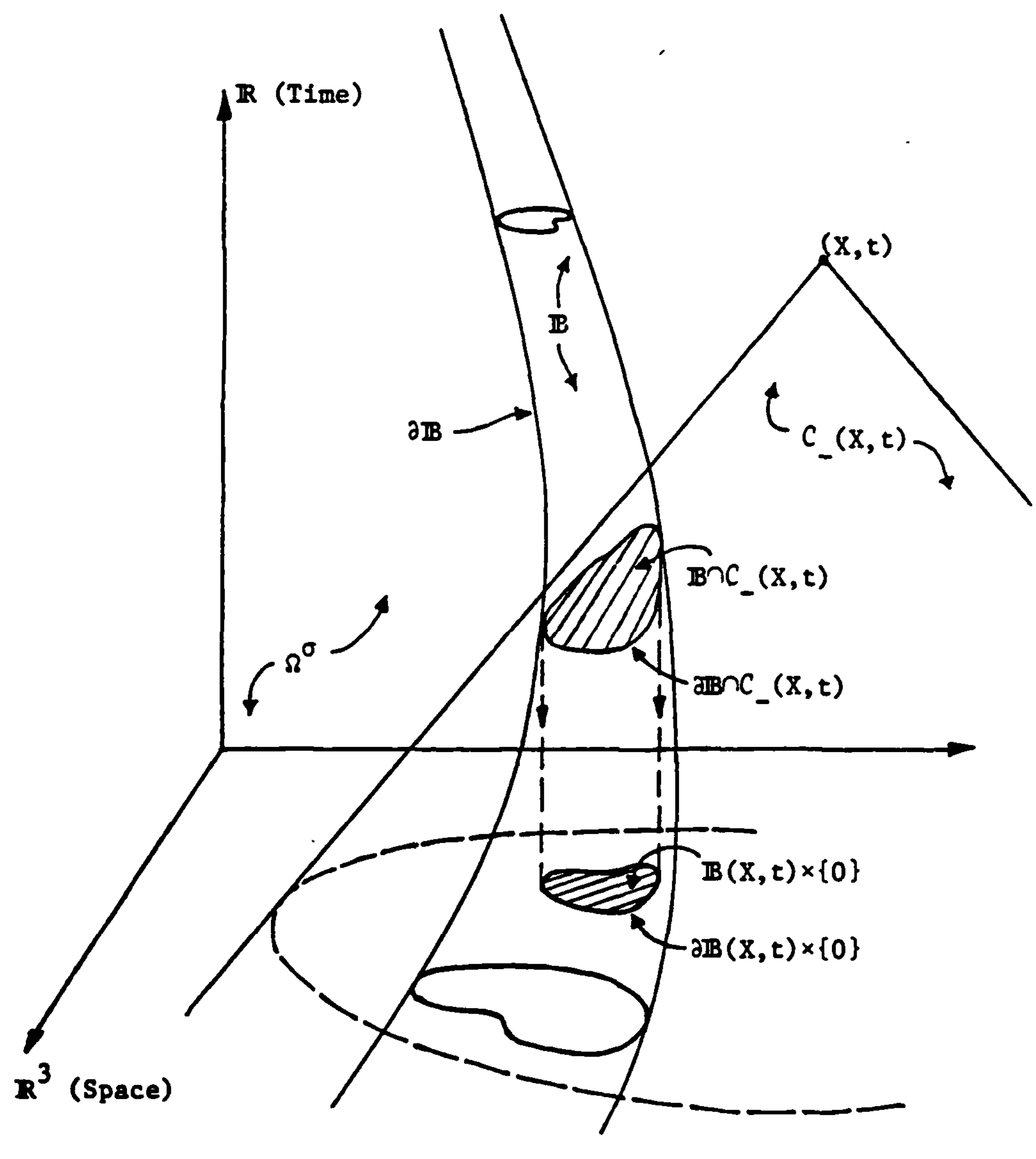

FIGURE 6. Retarded sets associated with a motion (cf., [V.18.i])

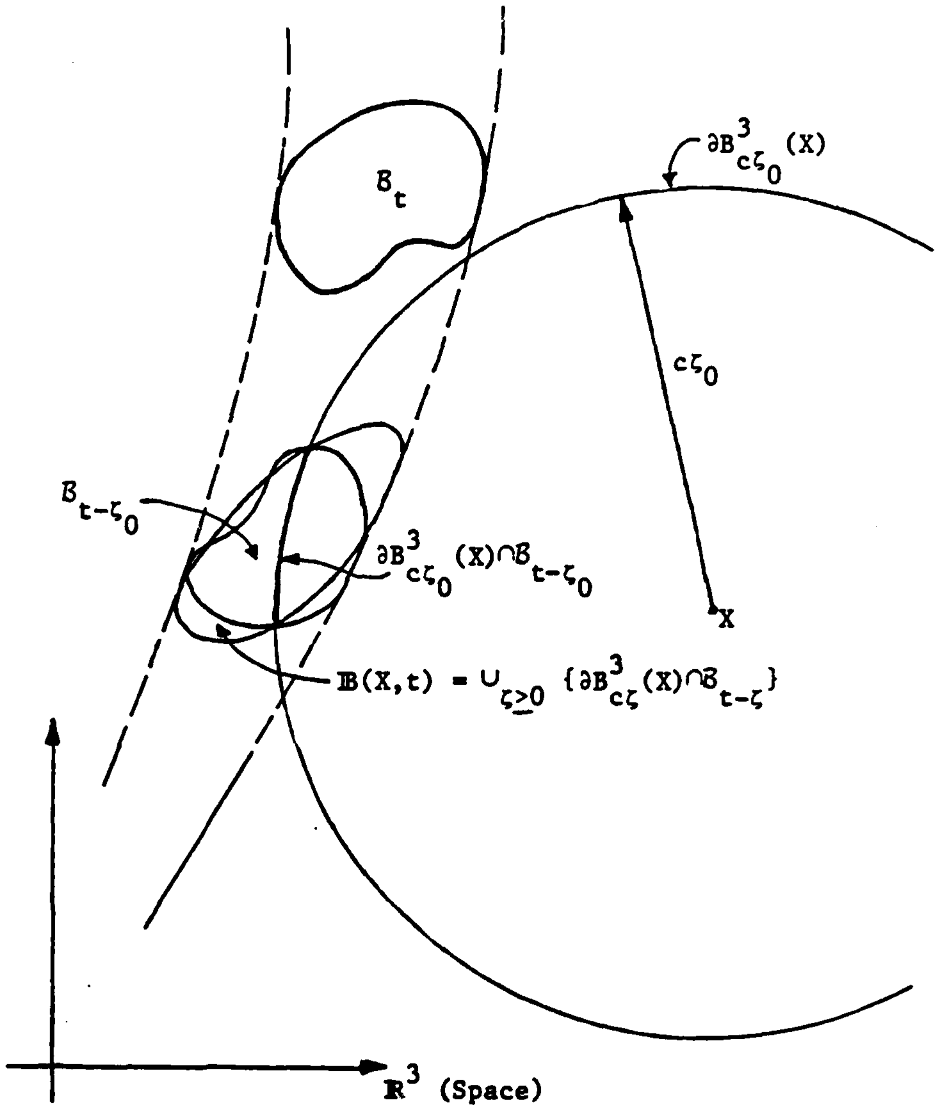

FIGURE 7. Generation of $\mathbb{B}(X,t)$ (cf., [V.18.iii])

(iii) $A(X,t) = \cup_{\zeta\geq 0} \{\partial B^3_{c\zeta}(X) \cap A_{t-\zeta}\}$, *where* $B^3_0(X) := \{X\}$.

PROOF. (i) If $Y \in A(X,t)$, then $(Y, t-\frac{1}{c} r_X(Y)) \in A$, by definition of $A(X,t)$, while clearly $(Y, t-\frac{1}{c} r_X(Y)) \in C_-(X,t)$. Since $Y = \mathrm{Pr}\,(Y, t-\frac{1}{c} r_X(Y))$, $Y \in \mathrm{Pr}\,(A\cap C_-(X,t))$. On the other hand, if $Y \in \mathrm{Pr}\,(A\cap C_-(X,t))$, then $(Y,\zeta) \in A\cap C_-(X,t)$ for some $\zeta \in \mathbb{R}$, which must be $t-\frac{1}{c} r_X(Y)$, since $(Y,\zeta) \in C_-(X,t)$. Thus, $(Y, t-\frac{1}{c} r_X(Y)) \in A$, so $Y \in A(X,t)$.

(ii) $Y \in A(X,t)$ iff $(Y, t-\frac{1}{c} r_X(Y)) \in A$, iff $Y \in A_{t-\frac{1}{c} r_X(Y)}$.

(iii) If $Y \in A(X,t)$, set $\zeta_Y := \frac{1}{c} r_X(Y)$. Then $Y \in \partial B^3_{c\zeta_Y}(X)$, and $Y \in A_{t-\frac{1}{c} r_X(Y)} = A_{t-\zeta_Y}$. Now, suppose $Y \in \partial B^3_{c\zeta}(X) \cap A_{t-\zeta}$, for some $\zeta \geq 0$. Then $r_X(Y) = c\zeta$, giving $\zeta = \frac{1}{c} r_X(Y)$, and so $Y \in A_{t-\frac{1}{c} r_X(Y)}$. Thus, $Y \in A(X,t)$. $\square$.

[V.19] PROPOSITION. *Let* $X \in \mathbb{R}^3$, $t \in \mathbb{R}$.

(i) *If* $A \subset \mathbb{R}^4$, *then* $A(X,t)' = A'(X,t)$, *so*

$$A(X,t)' = \mathrm{Pr}\,(A'\cap C_-(X,t)) = \left\{Y \in \mathbb{R}^3 \mid \ Y \in A'_{t-\frac{1}{c} r_X(Y)}\right\}$$

$$= \cup_{\zeta\geq 0} \{\partial B^3_{c\zeta}(X) \cap A'_{t-\zeta}\}.$$

(ii) *If* $\{A_\iota \mid \iota \in I\}$ *is any family of subsets of* $\mathbb{R}^4$, *then*

$$\{\cup_{\iota\in I} A_\iota\}(X,t) = \cup_{\iota\in I} A_\iota(X,t),$$

$$\{\cap_{\iota\in I} A_\iota\}(X,t) = \cap_{\iota\in I} A_\iota(X,t).$$

P R O O F. (i) Simply note that $A'(X,t) = P^{-1}_{(X,t)}(A') = \{P^{-1}_{(X,t)}(A)\}' = A(X,t)'$, and then the remaining statements of (i) follow from [V.18].

(ii) $\{\cup_{\iota\in I} A_\iota\}(X,t) = P^{-1}_{(X,t)}(\cup_{\iota\in I} A_\iota) = \cup_{\iota\in I} P^{-1}_{(X,t)}(A_\iota) = \cup_{\iota\in I} A_\iota(X,t)$. The second statement is proven in the same manner. □.

[V.20] P R O P O S I T I O N. *Let* $A \subset \mathbb{R}^4$, $X \in \mathbb{R}^3$, $t \in \mathbb{R}$.

(i) *If* A *is open [closed], then* $A(X,t)$ *is open [closed];*

(ii) $A(X,t)^- \subset A^-(X,t)$;

(iii) $A^\circ(X,t) \subset A(X,t)^\circ$;

(iv) $\partial\{A(X,t)\} \subset \{\partial A\}(X,t) = \cup_{\zeta\geq 0} \{\partial B^3_{c\zeta}(X)\cap(\partial A)_{t-\zeta}\}$.

P R O O F. (i), (ii), and (iii) follow from the definition $A(X,t) := P^{-1}_{(X,t)}(A)$ and the continuity of $P_{(X,t)}$, upon recalling the various well-known necessary and sufficient conditions for the continuity of a function from one topological space to another, in terms of inverse images.

(iv) The inclusion here is proven by using (ii), [V.19.i], and the second statement of [V.19.ii], writing

$$\partial\{A(X,t)\} = A(X,t)^{-}\cap A(X,t)'^{-} = A(X,t)^{-}\cap A'(X,t)^{-}$$

$$\subset A^{-}(X,t)\cap A'^{-}(X,t) = \{A^{-}\cap A'^{-}\}(X,t)$$

$$= \{\partial A\}(X,t).$$

The equality is simply an application of [V.18.iii]. $\square$.

It shall also turn out to be important to know how X is situated relative to $A(X,t)$, both set-theoretically and topologically. In general, we have the following facts concerning this question:

[V.21] P R O P O S I T I O N. *Let* $A \subset \mathbb{R}^4$, $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. *Then*

(i) $X \in A(X,t)$ *iff* $(X,t) \in A$;

(ii) $X \in A(X,t)^{\circ}$ *if* $(X,t) \in A^{\circ}$;

(iii) $X \in A(X,t)^{-\prime}$ *if* $(X,t) \in A^{-\prime}$;

(iv) $(X,t) \in \partial A$ *if* $X \in \partial\{A(X,t)\}$.

P R O O F. (i) $X \in A(X,t)$ iff $P_{(X,t)}(X) \in A$. But $P_{(X,t)}(X) = (X,t)$.

(ii) From (i), $(X,t) \in A^{\circ}$ implies that $X \in A^{\circ}(X,t) \subset A(X,t)^{\circ}$, the latter inclusion being just [V.20.iii].

(iii) Using (i), if $(X,t) \in A^{-\prime}$, then, since $A^{-\prime} = A'^{\circ}$, we find $X \in A'^{\circ}(X,t) \subset A'(X,t)^{\circ} = A(X,t)'^{\circ} = A(X,t)^{-\prime}$.

(iv) By [V.20.iv], $X \in \partial\{A(X,t)\}$ gives $X \in \{\partial A\}(X,t)$, so (i) shows that $(X,t) \in \partial A$. $\square$.

In general, statements stronger than [V.20.ii-iv] and [V.21.ii-iv] cannot be made, as simple counterexamples will show. We should like to identify a family of subsets of $\mathbb{R}^4$ for which the inclusions of [V.20.ii-iv] can be strengthened to equalities, and for which the converses of [V.21.ii-iv] are also true. Moreover, this family should include $\mathbb{B}$ and Ω^σ, for any motion M, and we shall demand that any retardation of a closed [open] set in the family be regularly closed [open]. The next definition identifies such a family.

[V.22] D E F I N I T I O N. Let $A \subset \mathbb{R}^4$. Then we say that A is of *type* (tℓ) iff whenever $X, Y \in \mathbb{R}^3$, $t \in \mathbb{R}$, and $(Y, t - \frac{1}{c} r_X(Y)) \in \partial A$, then $\{B^3_\varepsilon(Y) \times \mathbb{R}\} \cap C_-(X,t)$ meets both A° and A'° for each $\varepsilon > 0$. $\blacksquare$.

[V.23] R E M A R K S. The designation "type (tℓ)" is meant to be reminiscent of the term "time-like" (which already has a technical significance in the study of hyperbolic partial differential equations). In the retardation notation, the condition $(Y, t - \frac{1}{c} r_X(Y)) \in \partial A$ can be restated as just $Y \in \mathrm{Pr}\,(\partial A \cap C_-(X,t)) = \{\partial A\}(X,t)$. Since the definition is symmetric with respect to A and A', $A \subset \mathbb{R}^4$ is of type (tℓ) iff A' is of type (tℓ). $\mathbb{R}^4$ and $\emptyset$ satisfy the definition vacuously, since each has empty boundary. Otherwise, *i.e.*, if A is a non-void proper subset of $\mathbb{R}^4$, then in order for A to be of type (tℓ), it is certainly necessary that A° and A'° be non-void. Note also that if A is of type (tℓ) and $(X,t) \in \partial A$, then $\{B^3_\varepsilon(X) \times \mathbb{R}\} \cap C_-(X,t)$ meets both A° and A'° for each $\varepsilon > 0$, whence it easily follows that $(X,t) \in A^{\circ-} \cap A'^{\circ-}$; it is not surprising, then, that if A is open [closed], it is necessarily regularly open [closed]; cf., [V.24.v], *infra*.

Finally, for $X,Y \in \mathbb{R}^3$, $t \in \mathbb{R}$, and $\varepsilon > 0$, we record the obvious equality

$$\{B^3_\varepsilon(Y) \times \mathbb{R}\} \cap C_-(X,t) = \{(Z,\ t - \frac{1}{c} r_X(Z)) \mid\ Z \in \mathbb{R}^3,\ \ r_Y(Z) < \varepsilon\}.$$

[V.24] P R O P O S I T I O N. *Let* $A \subset \mathbb{R}^4$ *be of type* $(t\ell)$. *Choose* $X \in \mathbb{R}^3$ *and* $t \in \mathbb{R}$. *Then*

(*i*) $A^\circ(X,t) = A(X,t)^\circ$;

(*ii*) $\partial\{A(X,t)\} = \{\partial A\}(X,t) = \cup_{\zeta \geq 0} \{\partial B^3_{c\zeta}(X) \cap (\partial A)_{t-\zeta}\}$; *in particular, there is no ambiguity in the meaning of the symbol* "$\partial A(X,t)$";

(*iii*) $A^-(X,t) = A(X,t)^-$;

(*iv*) *if* A *is closed [open],* $A(X,t)$ *is regularly closed [open];*

(*v*) *if* A *is closed [open],* A *is regularly closed [open].*

P R O O F. (i) According to [V.20.iii], we need only prove the inclusion $A(X,t)^\circ \subset A^\circ(X,t)$. Then let $Y \in A(X,t)^\circ$. $B^3_\varepsilon(Y) \subset A(X,t)$ for some $\varepsilon > 0$, so $Z \in A(X,t)$, *i.e.*, $(Z,\ t - \frac{1}{c} r_X(Z)) \in A$, whenever $Z \in \mathbb{R}^3$ and $r_Y(Z) < \varepsilon$. That is,

$$\{B_{\varepsilon}^{3}(Y)\times\mathbb{R}\}\cap C_{-}(X,t) = \{(Z,\ t-\frac{1}{c}\,r_X(Z))\mid\ Z\in\mathbb{R}^3,\ \ r_Y(Z)<\varepsilon\}\subset A.$$

In particular, $\{B_{\varepsilon}^{3}(Y)\times\mathbb{R}\}\cap C_{-}(X,t)$ does not meet A'^{o}, so we cannot have $(Y,\ t-\frac{1}{c}\,r_X(Y))\in\partial A$, since A is of type $(t\ell)$. We do have $Y\in A(X,t)$, so $(Y,\ t-\frac{1}{c}\,r_X(Y))\in A$. Thus, $(Y,\ t-\frac{1}{c}\,r_X(Y))\in A\cap(\partial A)' = A^{o}$, so $Y\in A^{o}(X,t)$, and the proof of (i) is complete.

(ii) According to [V.20.iv], we need only prove the inclusion $\{\partial A\}(X,t)\subset\partial\{A(X,t)\}$. Suppose $Y\in\{\partial A\}(X,t)$, so that $(Y,\ t-\frac{1}{c}\,r_X(Y))\in\partial A$. If we assume that $Y\in A(X,t)^{o}$, then, just as in the reasoning of (i), we find $\{B_{\varepsilon}^{3}(Y)\times\mathbb{R}\}\cap C_{-}(X,t)\subset A$ for some $\varepsilon>0$, which is impossible, since A is of type $(t\ell)$ and $(Y,\ t-\frac{1}{c}\,r_X(Y))\in\partial A$. Similarly, the assumption that $Y\in A(X,t)'^{o} = A'(X,t)^{o}$ leads to the inclusion $\{B_{\delta}^{3}(Y)\times\mathbb{R}\}\cap C_{-}(X,t)\subset A'$ for some $\delta>0$, which is again an impossibility, since A is of type $(t\ell)$. Thus, we must have $Y\in A(X,t)^{o\prime}\cap A(X,t)'^{o\prime} = A(X,t)'^{-}\cap A(X,t)^{-} = \partial\{A(X,t)\}$.

(iii) Using (ii) and [V.19.ii], we can write

$$\begin{aligned} A^{-}(X,t) &= \{A\cup\partial A\}(X,t) = A(X,t)\cup\{\partial A\}(X,t)\\ &= A(X,t)\cup\partial\{A(X,t)\} = A(X,t)^{-}. \end{aligned}$$

(iv) Suppose A is closed. Then $A(X,t)$ is closed, so $A(X,t)^{o-}\subset A(X,t)$. To prove the reversed inclusion, let $Y\in A(X,t)$. We can suppose that $Y\in\partial\{A(X,t)\}$, for, $A(X,t) = A(X,t)^{o}\cup\partial\{A(X,t)\}$ (since $A(X,t)$ is closed) and the assumption $Y\in A(X,t)^{o}$ obviously leads to $Y \in A(X,t)^{o-}$. Now, by (ii), $Y \in \partial\{A(X,t)\} = \{\partial A\}(X,t)$, so $(Y, t-\frac{1}{c}r_X(Y)) \in \partial A$. But, A being of type $(t\ell)$, we then know that $\{B_\varepsilon^3(Y)\times\mathbb{R}\}\cap C_-(X,t) = \{(Z, t-\frac{1}{c}r_X(Z)) \mid Z \in \mathbb{R}^3, r_Y(Z) < \varepsilon\}$ meets A^o for each $\varepsilon > 0$. Clearly then, we can construct a sequence $(Z_i)_1^\infty \subset \mathbb{R}^3$ such that $\lim_{i\to\infty} Z_i = Y$ and $(Z_i, t-\frac{1}{c}r_X(Z_i)) \in A^o$ for each $i \in \mathbb{N}$, *i.e.*, $Z_i \in A^o(X,t) = A(X,t)^o$ for each $i \in \mathbb{N}$. The existence of such a sequence shows that $Y \in A(X,t)^{o-}$. Thus, $A(X,t)$ is regularly closed.

Now, suppose A is open. Then A' is closed and of type $(t\ell)$, so $A'(X,t)$ is regularly closed, by the first part of the proof. Then $A(X,t)'$ $(= A'(X,t))$ is regularly closed, whence $A(X,t)$ is regularly open.

(v) Observe that, if $A_1,A_2 \subset \mathbb{R}^4$, then $A_1 = A_2$ iff $A_1(Y,\xi) = A_2(Y,\xi)$ for each $(Y,\xi) \in \mathbb{R}^4$: this follows directly from [V.21.i]. Now suppose A is closed. For each $(Y,\xi) \in \mathbb{R}^4$, $A(Y,\xi)$ is regularly closed (by (iv)), so, using (i) and (iii), $A^{o-}(Y,\xi) = A(Y,\xi)^{o-} = A(Y,\xi)$. The equality $A^{o-} = A$ follows from the observation just made. The proof that A is regularly open whenever A is open is quite similar, so we omit it. □.

[V.25] P R O P O S I T I O N. *Let* $A \subset \mathbb{R}^4$ *be of type* $(t\ell)$, $X \in \mathbb{R}^3$, *and* $t \in \mathbb{R}$. *Then*

(i) $X \in \partial A(X,t)$ *iff* $(X,t) \in \partial A$;

(ii) $X \in A(X,t)^{o}$ *iff* $(X,t) \in A^{o}$;

(iii) $X \in A(X,t)^{-}$ *iff* $(X,t) \in A^{-}$;

(iv) $X \in A(X,t)^{-'}$ *iff* $(X,t) \in A^{-'}$.

P R O O F. These are immediate from [V.21.i] and the equalities $\partial\{A(X,t)\} = \{\partial A\}(X,t)$, $A(X,t)^{o} = A^{o}(X,t)$, $A(X,t)^{-} = A^{-}(X,t)$, and $A(X,t)^{-'} = A^{-'}(X,t)$; cf., [V.24] and [V.19.i]. $\square$.

We can give an alternate characterization of sets of type $(t\ell)$:

[V.26] P R O P O S I T I O N. *Let* $A \subset \mathbb{R}^4$. *Then* A *is of type* $(t\ell)$ *iff, whenever* $X \in \mathbb{R}^3$ *and* $t \in \mathbb{R}$,

$$\{\partial A\}(X,t) \subset A^{o}(X,t)^{-} \cap A'^{o}(X,t)^{-}.$$

P R O O F. Suppose A is of type $(t\ell)$. Choose $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. Let $Y \in \{\partial A\}(X,t)$, so $(Y,\ t-\frac{1}{c}\,r_X(Y)) \in \partial A$. Let $\varepsilon > 0$. We can find $(Z_{1\varepsilon},\zeta_{1\varepsilon})$ and $(Z_{2\varepsilon},\zeta_{2\varepsilon}) \in \mathbb{R}^4$ with

$$(Z_{1\varepsilon},\zeta_{1\varepsilon}) \in \{B^3_\varepsilon(Y)\times\mathbb{R}\} \cap C_-(X,t) \cap A^{o},$$

$$(Z_{2\varepsilon},\zeta_{2\varepsilon}) \in \{B^3_\varepsilon(Y)\times\mathbb{R}\} \cap C_-(X,t) \cap A'^{o}.$$

Clearly, $\zeta_{1\varepsilon} = t-\frac{1}{c}\,r_X(Z_{1\varepsilon})$ and $\zeta_{2\varepsilon} = t-\frac{1}{c}\,r_X(Z_{2\varepsilon})$, whence $Z_{1\varepsilon} \in B^3_\varepsilon(Y) \cap A^{o}(X,t)$, $Z_{2\varepsilon} \in B^3_\varepsilon(Y) \cap A'^{o}(X,t)$. Thus, each neighborhood of Y meets $A^{o}(X,t)$ and $A'^{o}(X,t)$, giving $Y \in A^{o}(X,t)^{-} \cap A'^{o}(X,t)^{-}$.[+]

[+] Observe that equality even holds in this case, for, by [V.24], $A^{o}(X,t)^{-} \cap A'^{o}(X,t)^{-} = A^{o-}(X,t) \cap A'^{o-}(X,t) \subset A^{-}(X,t) \cap A'^{-}(X,t) = \{A^{-} \cap A'^{-}\}(X,t) = \{\partial A\}(X,t)$. Of course, we also have $\{\partial A\}(X,t) = \partial\{A(X,t)\}$.

Conversely, suppose the inclusion holds whenever $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. Choose $X \in \mathbb{R}^3$, $t \in \mathbb{R}$, and suppose $Y \in \mathbb{R}^3$ satisfies $(Y, t-\frac{1}{c} r_X(Y)) \in \partial A$. Then $Y \in \{\partial A\}(X,t)$, so $Y \in A^o(X,t)^- \cap A'^o(X,t)^-$. Let $\varepsilon > 0$. Then we can find $Z_{1\varepsilon} \in B^3_\varepsilon(Y) \cap A^o(X,t)$, $Z_{2\varepsilon} \in B^3_\varepsilon(Y) \cap A'^o(X,t)$. It follows easily that

$$(Z_{1\varepsilon}, t-\frac{1}{c} r_X(Z_{1\varepsilon})) \in \{B^3_\varepsilon(Y)\times\mathbb{R}\} \cap C_-(X,t) \cap A^o,$$

$$(Z_{2\varepsilon}, t-\frac{1}{c} r_X(Z_{2\varepsilon})) \in \{B^3_\varepsilon(Y)\times\mathbb{R}\} \cap C_-(X,t) \cap A'^o.$$

Therefore, A fulfills the requirements of [V.22]. □.

Having [V.26], we can now show that Postulate [V.1.iv] yields the following important result:

[V.27] PROPOSITION. *Let M be a motion. Then the associated sets $\mathbb{B}$ and Ω^σ are of type* $(t\ell)$.

PROOF. Since $\Omega^\sigma := \mathbb{B}'$, it suffices to show that $\mathbb{B}$ is of type $(t\ell)$. Choose $X \in \mathbb{R}^3$ and $t \in \mathbb{R}$. According to [V.26], we must prove that

$$\{\partial\mathbb{B}\}(X,t) \subset \mathbb{B}^o(X,t)^- \cap \mathbb{B}'(X,t)^- \tag{1}$$

(recall that $\mathbb{B}'$ is open). Suppose first that M is a null motion, so that $\mathcal{B}_\zeta = \mathcal{B}_0$ for each $\zeta \in \mathbb{R}$, and $\mathbb{B} = \mathcal{B}_0 \times \mathbb{R}$. It is easy to see from [V.18.ii] that $\mathbb{B}^o(X,t) = \mathcal{B}^o_0$, since $(\mathbb{B}^o)_\zeta = \mathcal{B}^o_0$ for each $\zeta \in \mathbb{R}$. Similarly, $\mathbb{B}'(X,t) = \mathcal{B}'_0$, and $\{\partial\mathbb{B}\}(X,t) = \partial\mathcal{B}_0$. Thus,

$$\{\partial\mathbb{B}\}(X,t) = \partial\mathcal{B}_0 = \mathcal{B}_0 \cap \mathcal{B}_0'^- = \mathcal{B}_0^{o-} \cap \mathcal{B}_0'^- \doteq \mathbb{B}^o(X,t)^- \cap \mathbb{B}'(X,t)^-,$$

having used the fact that $\mathcal{B}_0$ is regularly closed. Thus, (1) holds in this case.

Assume now that $c^* > 0$. Suppose that $Z \in \{\partial\mathbb{B}\}(X,t)$, so we have $Z \in (\partial\mathbb{B})_{t-\frac{1}{c}r_X(Z)} = \partial\mathcal{B}_{t-\frac{1}{c}r_X(Z)}$. We shall first show that $Z \in \mathbb{B}^o(X,t)^-$: using [V.1.iv], choose $\varepsilon \in (0,2\eta(Z,\ t-\frac{1}{c}r_X(Z))]$, and let $p_{I,\varepsilon/2}\colon\ [-\varepsilon/2,\ \varepsilon/2] \to \mathbb{R}^3$ be a continuous function such that

$$p_{I,\varepsilon/2}(\zeta) \in B^3_{\varepsilon/2}(Z)^- \cap \mathcal{B}^o_{t-\frac{1}{c}r_X(Z)+\frac{1}{c}\zeta} \quad \text{whenever} \quad |\zeta| \le \varepsilon/2. \tag{2}$$

We define $f_{I,\varepsilon}\colon\ [-\varepsilon/2,\ \varepsilon/2] \to \mathbb{R}$ by

$$f_{I,\varepsilon}(\zeta) := r_X(Z)-r_X(p_{I,\varepsilon/2}(\zeta)) \quad \text{for} \quad |\zeta| \le \varepsilon/2. \tag{3}$$

Clearly, $|f_{I,\varepsilon}(\zeta)| \le |Z-p_{I,\varepsilon/2}(\zeta)|_3 \le \varepsilon/2$ if $|\zeta| \le \varepsilon/2$, by (2), showing that $f_{I,\varepsilon}([-\varepsilon/2,\ \varepsilon/2]) \subset [-\varepsilon/2,\ \varepsilon/2]$. The continuity of $f_{I,\varepsilon}$ follows from that of $p_{I,\varepsilon/2}$. Using a familiar fact, we can now assert that $f_{I,\varepsilon}$ possesses a fixed point $\zeta^* \in [-\varepsilon/2,\ \varepsilon/2]$. By (3), this number satisfies

$$-r_X(Z)+\zeta^* = -r_X(p_{I,\varepsilon/2}(\zeta^*)),$$

so that

$$p_{I,\varepsilon/2}(\zeta^*) \in \mathcal{B}^o_{t-\frac{1}{c}r_X(Z)+\frac{1}{c}\zeta^*} = \mathcal{B}^o_{t-\frac{1}{c}r_X(p_{I,\varepsilon/2}(\zeta^*))}$$

$$= (\mathbb{B}^o)_{t-\frac{1}{c}r_X(p_{I,\varepsilon/2}(\zeta^*))}.$$

In view of [V.18.ii], the latter inclusion implies that $p_{I,\varepsilon/2}(\zeta^*)$ lies in $\mathbb{B}^o(X,t)$. Thus,

$$p_{I,\varepsilon/2}(\zeta^*) \in B^3_{\varepsilon/2}(Z)^- \cap \mathbb{B}^o(X,t) \subset B^3_\varepsilon(Z) \cap \mathbb{B}^o(X,t).$$

We conclude that $B^3_\varepsilon(Z) \cap \mathbb{B}^o(X,t) \neq \emptyset$ for all sufficiently small positive ε, so $Z \in \mathbb{B}^o(X,t)^-$. By using the other half of Postulate [V.1.iv] in an analogous manner, one can show also that $Z \in \mathbb{B}'(X,t)^-$. Accordingly, (1) holds in all cases. With this, the proof is complete. □.

We shall digress here briefly from the major line of development, for the purpose of amplifying the remarks of [V.2.b]. Specifically, we intend to identify a condition which is at once reasonably weak, easily verified, and implies [V.1.iv].

[V.28] D E F I N I T I O N. A *finite cone in* $\mathbb{R}^n$ is a set of the form

$$B^n_h(x_V) \cap \{x_V + \lambda(y - x_V) \mid y \in B^n_\delta(x), \quad \lambda > 0\},$$

wherein $h > 0$, $\delta > 0$, and x_V and x are points of $\mathbb{R}^n$ with $|x - x_V|_n > \delta$; x_V is termed the *vertex* of the cone, h its *height*, and $\sin^{-1}(\delta/|x - x_V|_n)$ its *vertex half-angle*. ■.

[V.29] P R O P O S I T I O N. *Let* $(\{B_\zeta\}_{\zeta \in \mathbb{R}}, R, X)$ *be an ordered triple possessing the properties described in* [V.1.i, ii, *and* iii] *and also satisfying the following condition:*

> [V.1.iv]' *if* $c^* > 0$, *then whenever* $\zeta \in \mathbb{R}$ *and* $Z \in \partial B_\zeta$ *there exist finite cones* $C^I(Z,\zeta)$ *and* $C^E(Z,\zeta)$ *in* $\mathbb{R}^3$,

with common vertex at Z, *with vertex half-angles* $\theta^{I}(Z,\zeta)$ *and* $\theta^{E}(Z,\zeta)$, *respectively, satisfying*

$$\sin\theta^{i}(Z,\zeta) > c^{*}/c \quad \text{for} \quad i = I \text{ or } E \tag{1}$$

and such that $C^{I}(Z,\zeta) \subset B^{o}_{\zeta}$ *and* $C^{E}(Z,\zeta) \subset B'_{\zeta}$.

Then the ordered triple also fulfills the requirements of [V.1.iv], *whence it belongs to* $\mathbb{M}$ *and generates a motion.*

P R O O F. If $c^{*} = 0$, then [V.1.iv] follows, as we have already observed in [V.5.c]. Now, suppose that $c^{*} > 0$. Choose $\zeta \in \mathbb{R}$, then $Z \in \partial B_{\zeta}$, and let $C^{I}(Z,\zeta)$ and $C^{E}(Z,\zeta)$ denote finite cones in $\mathbb{R}^3$ as in [V.1.iv]'. Let $h^{I}(Z,\zeta)$ denote the height of $C^{I}(Z,\zeta)$, and select $\varepsilon \in (0, \frac{1}{2} h^{I}(Z,\zeta)]$. Let $\hat{Z}$ denote the point of $C^{I}(Z,\zeta)$ which lies on the axis of this cone at distance $|Z-\hat{Z}|_3 = \varepsilon$ from Z. Define $p_{I\varepsilon}$: $[-\varepsilon,\varepsilon] \to \mathbb{R}^3$ by simply setting

$$p_{I\varepsilon}(\xi) := \hat{Z} \quad \text{for} \quad |\xi| \le \varepsilon.$$

We shall show that $\hat{Z} \in B^{o}_{\zeta+\frac{1}{c}\xi}$ whenever $|\xi| \le \varepsilon$, whence it shall obviously follow that $p_{I\varepsilon}$ fulfills the requirements of [V.1.iv]. Because $\varepsilon \le \frac{1}{2} h^{I}(Z,\zeta)$, it is easy to see that

$$B^{3}_{\varepsilon\cdot\sin\theta^{I}(Z,\zeta)}(\hat{Z}) \subset C^{I}(Z,\zeta) \subset B^{o}_{\zeta},$$

so $\hat{Z} \in B^{o}_{\zeta}$, with

$$\operatorname{dist}(\hat{Z},\partial B_{\zeta}) \ge \varepsilon\cdot\sin\theta^{I}(Z,\zeta) > \varepsilon\cdot\frac{c^{*}}{c}, \tag{2}$$

the second inequality holding by (1). Now, recalling [V.12.i] (the validity of which derives from [V.1.i, ii, and iii] alone, as one can easily check), we find that

$$\hat{Z} \in \mathcal{B}^{o}_{\zeta+\frac{1}{c}\xi} \qquad \text{whenever} \qquad |\xi| < \frac{c}{c^*}\,\mathrm{dist}\,(\hat{Z},\partial\mathcal{B}_{\zeta}),$$

which, with (2), implies the desired result

$$\hat{Z} \in \mathcal{B}^{o}_{\zeta+\frac{1}{c}\xi} \qquad \text{whenever} \qquad |\xi| \leq \varepsilon.$$

Similarly, if we denote the height of $C^{E}(Z,\zeta)$ by $h^{E}(Z,\zeta)$, suppose that $\varepsilon \in (0, \frac{1}{2}\,h^{E}(Z,\zeta)]$, and take $\tilde{Z}$ to be the point of $C^{E}(Z,\zeta)$ lying on the axis of this cone with $|Z-\tilde{Z}|_3 = \varepsilon$, we can again use (1) to conclude that

$$\tilde{Z} \in \mathcal{B}'_{\zeta+\frac{1}{c}\xi} \qquad \text{whenever} \qquad |\xi| \leq \varepsilon.$$

Thus, the function $\xi \mapsto p_{E\varepsilon}(\xi) := \tilde{Z}$ on $[-\varepsilon,\varepsilon]$ fulfills the requirements of [V.1.iv]. We have now shown that, if $c^* > 0$, [V.1.iv] holds with $\eta(Z,\zeta) = \frac{1}{2}\min\{h^{I}(Z,\zeta),\, h^{E}(Z,\zeta)\}$, completing the proof. $\square$.

[V.30] R E M A R K. To carry the result of [V.29] over to the setting in which we shall later work exclusively, again let $(\{\mathcal{B}_{\zeta}\}_{\zeta\in\mathbb{R}},\mathcal{R},\mathcal{X})$ be an ordered triple as in [V.1.i, ii, and iii], but now assume, in place of [V.1.iv]', that $\partial\mathcal{B}_{\zeta}$ is a (2,3;1)-manifold for each $\zeta \in \mathbb{R}$. Then we can show that [V.1.iv]', and so also [V.1.iv], is fulfilled, implying that the triple belongs $\mathbb{M}$. In fact, let $\zeta \in \mathbb{R}$ and $Z \in \partial\mathcal{B}_{\zeta}$, and choose any $\theta \in (0, \pi/2)$: we shall show that there are finite cones $C^I(Z,\zeta) \subset B_\zeta^o$ and $C^E(Z,\zeta) \subset B_\zeta'$, each having vertex at Z and vertex half-angle equal to θ. Then, if $c^* > 0$, since $c^* < c$, we can surely select θ so that $\sin\theta > c^*/c$, and so find cones satisfying the requirements of [V.1.iv]'. To verify the claim, we note first that B_ζ^o is regularly open ([V.2.f]), so $\partial(B_\zeta^o) = \partial B_\zeta$, and [I.2.29] asserts that B_ζ^o is 1-regular. Thus, we can find a positive ρ and a function $\Phi \in C^1(B_\rho^3(Z))$ (each depending upon the particular $\zeta \in \mathbb{R}$ and $Z \in \partial B_\zeta$ which were chosen) such that grad Φ is non-vanishing in $B_\rho^3(Z)$,

$$\partial B_\zeta \cap B_\rho^3(Z) = \{\tilde{Y} \in B_\rho^3(Z) \mid \Phi(\tilde{Y}) = 0\},$$

and

$$B_\zeta^o \cap B_\rho^3(Z) = \{\tilde{Y} \in B_\rho^3(Z) \mid \Phi(\tilde{Y}) < 0\} \tag{1}$$

(with a corresponding equality describing $B_\zeta' \cap B_\rho^3(Z)$). Now, select a number $h \in (0,\rho]$ such that

$$|\text{grad } \Phi(\tilde{Y}) - \text{grad } \Phi(Z)|_3 < \cos\theta \cdot |\text{grad } \Phi(Z)|_3 \quad \text{for} \quad \tilde{Y} \in B_h^3(Z). \tag{2}$$

Writing $\nu_{\partial B_\zeta}(Z) := \nu_{\partial(B_\zeta^o)}(Z) = |\text{grad } \Phi(Z)|_3^{-1} \text{ grad } \Phi(Z)$ (cf., [I.2.32.b]), for each $\tilde{Y} \in \mathbb{R}^3 \cap \{Z\}'$ let $\theta_\nu(\tilde{Y}) := \cos^{-1}(|\tilde{Y}-Z|_3^{-1} \cdot (\tilde{Y}-Z) \bullet \nu_{\partial B_\zeta}(Z))$, *i.e.*, $\theta_\nu(\tilde{Y}) \in [0,\pi]$ and is the angle formed by $\nu_{\partial B_\zeta}(Z)$ and $(\tilde{Y}-Z)$. The set

$$C^I(Z,\zeta) := \{\tilde{Y} \in B_h^3(Z) \cap \{Z\}' \mid \theta_\nu(\tilde{Y}) \in (\pi-\theta, \pi]\}$$

is clearly a finite cone in $\mathbb{R}^3$ with vertex at Z, height h, and vertex half-angle θ; we now prove that $C^I(Z,\zeta) \subset B_\zeta^o$: for

$Y \in C^I(Z,\zeta)$, we can use the mean-value theorem, (2), and the obvious facts

$$\cos \theta_\nu(Y) < \cos(\pi-\theta) = -\cos\theta < 0$$

and $\Phi(Z) = 0$ to write, for some Y_Z lying between Y and Z on the straight line segment joining these points,

$$\begin{aligned}
\Phi(Y) &= \text{grad}\ \Phi(Y_Z) \bullet (Y-Z) \\
&\leq \text{grad}\ \Phi(Z) \bullet (Y-Z) + |\text{grad}\ \Phi(Y_Z) - \text{grad}\ \Phi(Z)|_3 \cdot |Y-Z|_3 \\
&= \cos\theta_\nu(Y) \cdot |\text{grad}\ \Phi(Z)|_3 \cdot |Y-Z|_3 \\
&\quad + |\text{grad}\ \Phi(Y_Z) - \text{grad}\ \Phi(Z)|_3 \cdot |Y-Z|_3 \\
&< \{-\cos\theta \cdot |\text{grad}\ \Phi(Z)|_3 + |\text{grad}\ \Phi(Y_Z) - \text{grad}\ \Phi(Z)|_3\} \cdot |Y-Z|_3 \\
&< 0.
\end{aligned}$$

This implies that $Y \in \mathcal{B}^o_\zeta$, in view of (1), completing the proof. A similar line of reasoning reveals that the finite cone

$$C^E(Z,\zeta) := \{\tilde{Y} \in B^3_h(Z) \cap \{Z\}' \mid \theta_\nu(\tilde{Y}) \in [0,\vartheta)\}$$

(with vertex at Z, height h, and vertex half-angle θ) lies in $\mathcal{B}'_\zeta$.

We return now to examine further implications of [V.1]. One of our principal goals is the study of the retardations of the sets $\mathbf{B}$ and Ω^σ associated with a motion M. Indispensable for this study are the existence and properties of the family of "retardation functions" generated by any motion, which we shall simply introduce here, intending to provide later a more detailed exposition of their characteristics.

[V.31] T H E O R E M. *Let* M *be a motion, and* (R,χ) *a reference pair for* M. *Corresponding to each* $P \in \partial R$, $X \in \mathbb{R}^3$, *and* $t \in \mathbb{R}$, *there exists a unique non-negative number* $\tau(P;X,t)$ *with the property*

$$r_X(\chi(P,\ t-\tau(P;X,t))) = c\tau(P;X,t).$$

P R O O F. Fix $P \in \partial R$, $X \in \mathbb{R}^3$, and $t \in \mathbb{R}$. Define an associated function F: $[0,\infty) \to [0,\infty)$ according to $F(\tau) := \frac{1}{c} r_X(\chi(P,\ t-\tau))$ for each $\tau \geq 0$. If τ_1 and τ_2 are non-negative,

$$|F(\tau_2)-F(\tau_1)| \leq \frac{1}{c}\ |\chi(P,\ t-\tau_2)-\chi(P,\ t-\tau_1)|_3 \leq \frac{c^*}{c}\ |\tau_2-\tau_1|,$$

whence it follows (since $c^* < c$) that F is a contraction on (the complete metric space) $[0,\infty)$ into itself. Banach's contraction mapping principle therefore yields the existence of a unique fixed point $\tau(P;X,t)$ for F, *i.e.*, a unique $\tau(P;X,t) \geq 0$ for which $F(\tau(P;X,t)) = \tau(P;X,t)$. This is just the assertion of the theorem. □.

We can now supply a catalogue of properties of the sets $\mathbb{B}(X,t)$ and $\Omega^\sigma(X,t)$, for $X \in \mathbb{R}^3$, $t \in \mathbb{R}$, associated with a motion M.

[V.32] T H E O R E M. *Let* M *be a motion. Choose* $X \in \mathbb{R}^3$ *and* $t \in \mathbb{R}$.

(*i*) $\mathbb{B}(X,t)$ *is non-void, compact, and regularly closed;*

(*ii*) $\Omega^\sigma(X,t) = \mathbb{B}(X,t)'$; $\Omega^\sigma(X,t)$ *is non-void and regularly open;*

(*iii*) $\partial\{\mathbb{B}(X,t)\} = \{\partial\mathbb{B}\}(X,t) = \cup_{\zeta \geq 0}\ \{\partial B^3_{c\zeta}(X) \cap \partial B_{t-\zeta}\}$;

(*iv*) $\mathbb{B}(X,t)^{o} = \mathbb{B}^{o}(X,t) = \cup_{\zeta \geq 0} \{\partial B^3_{c\zeta}(X) \cap B^{o}_{t-\zeta}\}$;

(*v*) $\Omega^{\sigma}(X,t)^{-} = \Omega^{\sigma -}(X,t) = \cup_{\zeta \geq 0} \{\partial B^3_{c\zeta}(X) \cap B'^{-}_{t-\zeta}\}$;

(*vi*) $X \in \mathbb{B}(X,t)$ *iff* $X \in B_t$; $X \in \Omega^{\sigma}(X,t)$ *iff* $X \in B'_t$;

(*vii*) $X \in \mathbb{B}(X,t)^{o}$ *iff* $X \in B^{o}_t$;

(*viii*) $X \in \partial\mathbb{B}(X,t)$ *iff* $X \in \partial B_t$;

(*ix*) $\mathbb{B}(X,t)$ *and* $\Omega^{\sigma}(X,t)$ *are intrinsic to* M.

P R O O F. (i) Let (R,χ) be a reference pair for M. Choose any $P \in \partial R$, and let $\tau(P;X,t)$ be as in [V.31]. Then

$$\chi(P,\ t-\tau(P;X,t)) \in B_{t-\tau(P;X,t)} = \mathbb{B}_{t-\tau(P;X,t)}$$

$$= \mathbb{B}_{t-\frac{1}{c} r_X(\chi(P,\ t-\tau(P;X,t)))};$$

with [V.18.ii], it clearly follows that $\chi(P,\ t-\tau(P;X,t)) \in \mathbb{B}(X,t)$. Consequently, we have shown that $\mathbb{B}(X,t) \neq \emptyset$. Next, since $\mathbb{B}$ is closed, $\mathbb{B}(X,t)$ is regularly closed, by [V.24.iv]. We must show then that $\mathbb{B}(X,t)$ is bounded. For this, note that, since B_t is bounded, there is some $s_0 > 0$ for which $B^3_{\hat{c}s}(B_t) \subset B^3_{cs}(X)$ whenever $s > s_0$; here, $\hat{c}$ has been chosen in (c^*,c). In fact, we can take

$$s_0 = \frac{1}{c-\hat{c}} \{\mathrm{dist}\ (X,B_t) + \mathrm{diam}\ B_t\},$$

since then, $s > s_0$ implies $cs > \hat{c}s + \mathrm{dist}\ (X,B_t) + \mathrm{diam}\ B_t$, and, supposing that $Y \in B^3_{\hat{c}s}(B_t)$, we find, using (I.1.4.1),

$$r_X(Y) \leq \text{dist}\,(Y,\mathcal{B}_t) + \text{dist}\,(X,\mathcal{B}_t) + \text{diam}\,\mathcal{B}_t < cs.$$

Thus, $Y \in B^3_{cs}(X)$ whenever $Y \in B^3_{\hat{c}s}(\mathcal{B}_t)$ and $s > s_0$, verifying the assertion. Now, according to [V.11.i], $\mathcal{B}_{t-s} \subset B^3_{\hat{c}s}(\mathcal{B}_t)$ if $s > 0$. Consequently, if $s > s_0$ we have $\mathcal{B}_{t-s} \subset B^3_{\hat{c}s}(\mathcal{B}_t) \subset B^3_{cs}(X)$, implying that $\partial B^3_{cs}(X) \cap \mathcal{B}_{t-s} = \emptyset$ for $s > s_0$. Finally, use of [V.18.iii] yields

$$\mathbb{B}(X,t) = \cup_{s \geq 0} \{\partial B^3_{cs}(X) \cap \mathbb{B}_{t-s}\} = \cup_{s \geq 0} \{\partial B_{cs}(X) \cap \mathcal{B}_{t-s}\}$$

$$= \cup_{0 \leq s \leq s_0} \{\partial B^3_{cs}(X) \cap \mathcal{B}_{t-s}\} \subset B^3_{cs_0}(X)^-,$$

so $\mathbb{B}(X,t)$ is bounded.

(ii) Since $\Omega^\sigma = \mathbb{B}'$, [V.19.i] shows that $\Omega^\sigma(X,t) = \mathbb{B}'(X,t) = \mathbb{B}(X,t)'$. The boundedness of $\mathbb{B}(X,t)$ shows that $\Omega^\sigma(X,t) \neq \emptyset$, while the conclusion that $\Omega^\sigma(X,t)$ is regularly open follows from either [V.24.iv] (since Ω^σ is open) or the fact that $\mathbb{B}(X,t)$ is regularly closed.

(iii), (iv), (v) All of (iii) and the first equalities in (iv) and (v) are immediate from [V.24.i, ii, iii], since $\mathbb{B}$ and Ω^σ are of type ($t\ell$) ([V.27]). The equalities $(\mathbb{B}^o)_\zeta = \mathcal{B}^o_\zeta$, $(\Omega^{\sigma-})_\zeta = \mathcal{B}'^-_\zeta$, for each $\zeta \in \mathbb{R}$, follow from [V.15.ii]; using these in conjunction with [V.18.iii] produces the second equalities in (iv) and (v).

(vi) By [V.21.i], $X \in \mathbb{B}(X,t)$ iff $(X,t) \in \mathbb{B}$, *i.e.*, iff $X \in \mathbb{B}_t = \mathcal{B}_t$. Thus, $X \in \Omega^\sigma(X,t) = \mathbb{B}(X,t)'$ iff $X \in \mathcal{B}'_t$.

(vii) This follows from (iv). Alternately, since $\mathbb{B}$ is of type $(t\ell)$, $X \in \mathbb{B}(X,t)^o$ iff $X \in \mathbb{B}^o(X,t)$, iff $(X,t) \in \mathbb{B}^o$, iff $X \in (\mathbb{B}^o)_t = \mathcal{B}^o_t$.

(viii) This statement follows just as that in (vii) (the equality $(\partial\mathbb{B})_t = \partial\mathcal{B}_t$ is implied by [V.15.iii]).

(ix) Recall that the positions $\mathcal{B}_\zeta$, $\zeta \in \mathbb{R}$, are intrinsic to M. Since $\mathbb{B}(x,t) = \cup_{\zeta\geq 0}\{\partial B^3_{c\zeta}(X)\cap\mathcal{B}_{t-\zeta}\}$ and $\Omega^\sigma(X,t) = \cup_{\zeta\geq 0}\{\partial B^3_{c\zeta}(X)\cap\mathcal{B}'_{t-\zeta}\}$ it is clear that $\mathbb{B}(X,t)$ and $\Omega^\sigma(X,t)$ are intrinsic to M. $\square$.

[V.33] R E M A R K. Let M be a null motion. In this case, all of the retardations of $\mathbb{B}$ and Ω^σ, which are now cylinders, are easily described. In fact, $\mathbb{B}_\zeta = \mathcal{B}_\zeta = \mathcal{B}_0$, and $\Omega^\sigma_\zeta = \mathcal{B}'_\zeta = \mathcal{B}'_0$, for each $\zeta \in \mathbb{R}$. From either [V.18.ii] or [V.18.iii], it is then clear that $\mathbb{B}(X,t) = \mathcal{B}_0$ and $\Omega^\sigma(X,t) = \mathcal{B}'_0$, for every choice of $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. In view of this, the statements of the just-proven Theorem [V.32] are reduced to trivialities in the case of a null motion. The retardation functions acquire a particularly simple form in this instance, as well. For example, taking the reference pair $(\mathcal{B}_0,\chi^o)$ for M, where $\chi^o(\cdot,\zeta)$: $\partial\mathcal{B}_0 \to \partial\mathcal{B}_0$ is the identity for each $\zeta \in \mathbb{R}$, it is obvious that the corresponding retardation function, τ^o, is given by $\tau^o(P;X,t) = \frac{1}{c}\, r_X(P)$, for each $P \in \partial\mathcal{B}_0$, $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. It turns out that these circumstances allow an investigation of scattering by stationary bodies which is, in all senses, simpler than the analysis required in the case of a non-null motion.

[V.34] R E M A R K. It was noted that a knowledge of the properties of the retardation functions associated with a motion is essential for the constructions to be undertaken later. We begin here with a description of the characteristics of these functions.

Recall, then, that if $\mathcal{M}$ is a motion, with any reference pair $(\mathcal{R},\chi)$ for $\mathcal{M}$ there is associated *via* [V.31] a unique "retardation" function[†] $\tau\colon\ \partial\mathcal{R}\times\mathbb{R}^3\times\mathbb{R}\to[0,\infty)$, defined implicitly by the requirement

$$\begin{gathered} r_X(\chi(P,\ t-\tau(P;X,t))) = c\tau(P;X,t), \\ \text{for each} \quad P\in\partial\mathcal{R}, \quad X\in\mathbb{R}^3, \quad t\in\mathbb{R}. \end{gathered} \tag{1}$$

Suppose that we think of the points of $\partial\mathcal{R}$ as "the particles" comprising the surface of the moving body at each instant: the path traced by the particle $P\in\partial\mathcal{R}$ is described by the function $\chi(P,\cdot)\colon \mathbb{R}\to\mathbb{R}^3$. Choose $X\in\mathbb{R}^3$, $t\in\mathbb{R}$, and suppose that a spherical wave-front emanates from X at time t, travelling with speed c. Let P retrace all its previous positions in reverse order, starting from $\chi(P,t)$ at time t (*i.e.*, "run the movie film backward, at the same speed"). Then $\tau(P;X,t)$ is the duration of time required for the spherical wave to intercept the particle, overtaking it at the position $\chi(P,\ t-\tau(P;X,t))$.

We might note that τ itself is not intrinsic to $\mathcal{M}$, but each range $\{\tau(P;X,t)\mid\ P\in\partial\mathcal{R}\}$, for $X\in\mathbb{R}^3$, $t\in\mathbb{R}$, is an intrinsic

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† The notation "τ" omits any indication of the dependence on the particular reference pair with which it is associated; this should cause no confusion.

object.

[V.35] P R O P O S I T I O N. *Let* M *be a motion,* (R,χ) *a reference pair for* M, *and* $\tau\colon \partial R\times\mathbb{R}^3\times\mathbb{R} \to [0,\infty)$ *the associated retardation function.*

(i) $\tau(P;X,t) = 0$ *for some* $P \in \partial R$, $X \in \mathbb{R}^3$, $t \in \mathbb{R}$ *iff* $X = \chi(P,t)$;

(ii) τ *is continuous;*

(iii) if $\chi\colon \partial R\times\mathbb{R} \to \mathbb{R}^3$ *is continuous in its first argument, uniformly in its second, then* τ *is uniformly continuous;*

(iv) for each fixed $P \in \partial R$, $\tau(P,\cdot,\cdot)$ *is Lipschitz continuous on* $\mathbb{R}^4$, *uniformly in* P; *in fact, if* $(X_1,t_1), (X_2,t_2) \in \mathbb{R}^4$,

$$|\tau(P;X_2,t_2)-\tau(P;X_1,t_1)| \leq \frac{1}{c-c^*}\{|X_2-X_1|_3+c^*|t_2-t_1|\}. \tag{1}$$

P R O O F. (i) Let $P \in \partial R$, $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. If $\tau(P;X,t) = 0$, then $r_X(\chi(P,t)) = 0$, *i.e.*, $X = \chi(P,t)$. Conversely, if $X = \chi(P,t)$, then $\hat{\tau} = 0$ is a solution of the equation $r_X(\chi(P,t-\hat{\tau})) = c\hat{\tau}$. Since $\tau(P;X,t)$ is the unique solution of this equation, we must have $\tau(P;X,t) = 0$.

(ii) Suppose $P_i \in \partial R$, $X_i \in \mathbb{R}^3$, and $t_i \in \mathbb{R}$, for $i = 1,2$: then

$$|\tau(P_2;X_2,t_2)-\tau(P_1;X_1,t_1)|$$

$$= \frac{1}{c}\,|r_{X_2}(\chi(P_2,t_2-\tau(P_2;X_2,t_2)))-r_{X_1}(\chi(P_1,t_1-\tau(P_1;X_1,t_1)))|$$

$$\le \frac{1}{c}\,|r_{X_2}(\chi(P_2,t_2-\tau(P_2;X_2,t_2)))-r_{X_2}(\chi(P_1,t_1-\tau(P_1;X_1,t_1)))|$$

$$+ \frac{1}{c}\,|r_{X_2}(\chi(P_1,t_1-\tau(P_1;X_1,t_1)))-r_{X_1}(\chi(P_1,t_1-\tau(P_1;X_1,t_1)))|$$

$$\le \frac{1}{c}\,|\chi(P_2,t_2-\tau(P_2;X_2,t_2))-\chi(P_1,t_1-\tau(P_1;X_1,t_1))|_3+\frac{1}{c}\,|X_2-X_1|_3$$

$$\le \frac{1}{c}\,|\chi(P_2,t_2-\tau(P_2;X_2,t_2))-\chi(P_2,t_1-\tau(P_1;X_1,t_1))|_3$$

$$+ \frac{1}{c}\,|\chi(P_2,t_1-\tau(P_1;X_1,t_1))-\chi(P_1,t_1-\tau(P_1;X_1,t_1))|_3+\frac{1}{c}\,|X_2-X_1|_3$$

$$\le \frac{1}{c}\,|\chi(P_2,t_1-\tau(P_1;X_1,t_1))-\chi(P_1,t_1-\tau(P_1;X_1,t_1))|_3$$

$$+ \frac{c^*}{c}\,\{|t_2-t_1|+|\tau(P_2;X_2,t_2)-\tau(P_1;X_1,t_1)|\}+\frac{1}{c}\,|X_2-X_1|_3,$$

from which we find

$$|\tau(P_2;X_2,t_2)-\tau(P_1;X_1,t_1)|$$

$$\le \frac{1}{c-c^*}\,\{|\chi(P_2,t_1-\tau(P_1;X_1,t_1))-\chi(P_1,t_1-\tau(P_1;X_1,t_1))|_3 \qquad (2)$$

$$+|X_2-X_1|_3+c^*|t_2-t_1|\}.$$

Since the function $P \mapsto \chi(P,t_1-\tau(P_1;X_1,t_1))$ is continuous on ∂R, the continuity of τ at (P_1,X_1,t_1) follows from the latter inequality. Thus, τ is continuous.

(iii) The hypothesis here is, more specifically: given $P \in \partial R$ and $\varepsilon > 0$, there is a $\delta(P,\varepsilon) > 0$ for which

$|\chi(Q,\zeta)-\chi(P,\zeta)|_3 < \varepsilon$ whenever $Q \in \partial R \cap B^3_{\delta(P,\varepsilon)}(P)$ and $\zeta \in \mathbb{R}$. From the compactness of ∂R, it follows easily that χ is then uniformly continuous in its first argument, uniformly in its second: given $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that $|\chi(P_2,\zeta)-\chi(P_1,\zeta)|_3 < \varepsilon$ whenever $P_1, P_2 \in \partial R$ with $|P_2-P_1|_3 < \delta_\varepsilon$ and $\zeta \in \mathbb{R}$. In view of inequality (2), the uniform continuity of τ follows.

Note that if χ is uniformly continuous on $\partial R \times \mathbb{R}$, then it satisfies the hypothesis of (iii).

(iv) Set $P_1 = P_2 = P$ in (2) to obtain (1). Then

$$|\tau(P;X_2,t_2)-\tau(P;X_1,t_1)| \leq \frac{\{1+(c^*)^2\}^{1/2}}{c-c^*} |(X_2,t_2)-(X_1,t_1)|_4, \tag{3}$$

i.e., $\tau(P;\cdot,\cdot)$ is Lipschitz continuous on $\mathbb{R}^4$, uniformly in $P \in \partial R$. □.

In the following definition, we introduce the notion of the "retardation" of a function defined on a subset of $\mathbb{R}^4$, with respect to a point in $\mathbb{R}^4$; we do this in two settings.

[V.36] D E F I N I T I O N S. (i) Let $A \subset \mathbb{R}^4$ be non-void, and f be any function on A. Let $X \in \mathbb{R}^3$ and $t \in \mathbb{R}$. If $A(X,t) \neq \emptyset$, then we define the *retardation of* f *with respect to* (X,t) to be the function $f \circ (P_{(X,t)}|\ A(X,t))$, on $A(X,t)$, and denote it by $[f]_{[X,t]}$. Thus, $[f]_{[X,t]}$ is given on $A(X,t)$ by

$$[f]_{[X,t]}(Y) := f(P_{(X,t)}(Y)) = f(Y,\ t-\tfrac{1}{c}\,r_X(Y)), \quad \text{for} \quad Y \in A(X,t). \tag{1}$$

Since $Y \in A(X,t)$ iff $P_{(X,t)}(Y) \in A$, it is clear that $[f]_{[X,t]}$ is well-defined, and is defined on $A(X,t) \subset \mathbb{R}^3$.

(ii) Let M be a motion, (R,χ) a reference pair for M, and τ the retardation function generated by (R,χ). Suppose f is a function on $\partial R \times \mathbb{R}$. If $X \in \mathbb{R}^3$, $t \in \mathbb{R}$, the *retardation of* f *with respect to* (X,t)† is defined to be the function $[f]_{(X,t)}$ on ∂R which is given by

$$[f]_{(X,t)}(P) := f(P,\ t-\tau(P;X,t)), \qquad \text{for} \qquad P \in \partial R. \qquad \blacksquare. \tag{2}$$

[V.37] R E M A R K S. (a) With notation as in [V.36.i], let A be of type $(t\ell)$. We wish to emphasize the fact that the location of X itself relative to the domain of $[f]_{[X,t]}$, $A(X,t)$, depends upon the situation of (X,t) relative to A: that is, $X \in A(X,t)$ iff $(X,t) \in A$, $X \in A(X,t)^{\circ}$ iff $(X,t) \in A^{\circ}$, $X \in \partial A(X,t)$ iff $(X,t) \in \partial A$, *etc.*, because of [V.21.i] and [V.25].

(b) Let M be a motion, and suppose f is defined on one of the sets $\mathbb{B}$, $\mathbb{B}^{\circ}$, Ω^{σ}, or $\partial\mathbb{B}$, with values in $\mathbb{R}^n$. Suppose $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. Then, as the case may be, $[f]_{[X,t]}$ is defined on $\mathbb{B}(X,t)$, $\mathbb{B}^{\circ}(X,t)$, $\Omega^{\sigma}(X,t)$, or $\partial\mathbb{B}(X,t)$, with values in $\mathbb{R}^n$. These will be the situations of most frequent interest. We emphasize that the domain of $[f]_{[X,t]}$ is a subset of $\mathbb{R}^3$.

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†Although the same terminology is used in both settings (i) and (ii), no confusion should arise, since different symbols are employed in the two cases.

(c) Let M, (R,χ), and τ have their usual meanings. Then χ: $\partial R \times \mathbb{R} \to \mathbb{R}^3$, so $[\chi]_{(X,t)}$: $\partial R \to \mathbb{R}^3$ is defined by [V.36.ii], whenever $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. Explicitly,

$$[\chi]_{(X,t)}(P) = \chi(P, t-\tau(P;X,t)), \quad \text{for} \quad P \in \partial R, \quad X \in \mathbb{R}^3, \quad t \in \mathbb{R}. \tag{1}$$

Consequently, the condition defining τ, (V.34.1), can be rewritten as

$$r_X \circ [\chi]_{(X,t)}(P) = c\tau(P;X,t), \quad \text{for} \quad P \in \partial R, \quad X \in \mathbb{R}^3, \quad t \in \mathbb{R}. \tag{2}$$

(d) On occasion, it will be convenient to employ the notation $[f](P;X,t)$ for either $[f]_{[X,t]}(P)$ as in [V.36.i] or $[f]_{(X,t)}(P)$ as in [V.36.ii]. Due regard for the context should suffice to clarify the meaning in any situation.

(e) The question of continuity of any of these retarded functions is certainly easily settled. With notation as in [V.36.i], if f: $A \to \mathbb{R}^n$ is continuous, then $[f]_{[X,t]}$: $A(X,t) \to \mathbb{R}^n$ is continuous, since $P_{(X,t)}$ is continuous. With notation as in [V.36.ii], if f: $\partial R \times \mathbb{R} \to \mathbb{R}^n$ is continuous, then $[f]_{(X,t)}$: $\partial R \to \mathbb{R}^n$ is continuous, being the composition of f with the continuous map $P \mapsto (P, t-\tau(P;X,t))$ on ∂R into $\partial R \times \mathbb{R}$ (the continuity of $\tau(\cdot;X,t)$ following from [V.35]).

The following fact is fundamental.

[V.38] P R O P O S I T I O N. *Let* M *be a motion,* (R,χ) *a reference pair for* M, *and* τ *the associated retardation function.*

Let $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. *Then* $[\chi]_{(X,t)}$: $\partial R \to \mathbb{R}^3$ *is a homeomorphism of* ∂R *onto* $\partial\mathbb{B}(X,t)$.

P R O O F. χ: $\partial R\times\mathbb{R} \to \mathbb{R}^3$ is continuous, so the continuity of $[\chi]_{(X,t)}$ follows, as a special case of the results of [V.37.e]. Now, suppose $P_1, P_2 \in \partial R$, and $[\chi]_{(X,t)}(P_1) = [\chi]_{(X,t)}(P_2)$. From (V.35.2), we find that $\tau(P_1;X,t) = \tau(P_2;X,t)$; denoting the common value by $\hat{\tau}$, then $\chi(P_1,t-\hat{\tau}) = \chi(P_2,t-\hat{\tau})$. Since $\chi(\cdot,t-\hat{\tau})$ is injective, we conclude that $P_1 = P_2$. Thus, $[\chi]_{(X,t)}$ is an injection.

Next, we show that $[\chi]_{(X,t)}(\partial R) = \partial\mathbb{B}(X,t)$: recall ([V.32.iii]) that $\partial\mathbb{B}(X,t)$ $(:= \partial\{\mathbb{B}(X,t)\} = \{\partial\mathbb{B}\}(X,t)) = \cup_{\zeta\geq 0} \{\partial B^3_{c\zeta}(X)\cap\partial\mathcal{B}_{t-\zeta}\}$, so $Y \in \partial\mathbb{B}(X,t)$ iff $Y \in \partial B^3_{c\zeta}(X)\cap\partial\mathcal{B}_{t-\zeta}$ for some $\zeta \geq 0$, whence it is clear that $Y \in \partial\mathbb{B}(X,t)$ iff $Y \in \partial\mathcal{B}_{t-\frac{1}{c}r_X(Y)}$. But, whenever $P \in \partial R$, $[\chi]_{(X,t)}(P) = \chi(P, t-\tau(P;X,t)) \in \partial\mathcal{B}_{t-\tau(P;X,t)} = \partial\mathcal{B}_{t-\frac{1}{c}r_X([\chi]_{(X,t)}(P))}$, so $[\chi]_{(X,t)}(P) \in \partial\mathbb{B}(X,t)$ for each $P \in \partial R$. On the other hand, suppose that $Y \in \partial\mathbb{B}(X,t)$. Then $Y \in \partial\mathcal{B}_{t-\frac{1}{c}r_X(Y)} = \chi(\partial R, t-\frac{1}{c}r_X(Y))$, so $Y = \chi(P_Y, t-\frac{1}{c}r_X(Y))$ for some $P_Y \in \partial R$. Obviously then, $c\cdot\frac{1}{c}r_X(Y) = r_X(\chi(P_Y, t-\frac{1}{c}r_X(Y)))$, from which we infer that $\frac{1}{c}r_X(Y) = \tau(P_Y;X,t)$. Finally, $Y = \chi(P_Y, t-\tau(P_Y;X,t)) = [\chi]_{(X,t)}(P_Y)$, *i.e.*, $Y \in [\chi]_{(X,t)}(\partial R)$. These facts show that $[\chi]_{(X,t)}(\partial R) = \partial\mathbb{B}(X,t)$, as claimed.

Gathering the results to this point, it has been shown that $[\chi]_{(X,t)}$ is a continuous bijection of ∂R onto $\partial\mathbb{B}(X,t)$. But the compactness of ∂R (and the fact that $\partial\mathbb{B}(X,t)$ is Hausdorff, of course) implies the continuity of $[\chi]^{-1}_{(X,t)}: \partial\mathbb{B}(X,t) \to \partial R$. This completes the proof. $\square$.

[V.39] N O T A T I O N. Recall that, for $\zeta \in \mathbb{R}$, we write χ_ζ^{-1}: $\partial B_\zeta \to \partial R$ for the inverse of the homeomorphism $\chi_\zeta := \chi(\cdot,\zeta)$ taking ∂R onto ∂B_ζ, whenever M is a motion and (R,χ) is a reference pair for M. Now we know that $\partial\mathbb{B} = \cup_{\zeta\geq 0} \{\partial B_\zeta \times\{\zeta\}\}$, so Corollary [V.8] states that $(Z,\zeta) \mapsto \chi_\zeta^{-1}(Z)$ is continuous from $\partial\mathbb{B}$ onto ∂R; since this map is defined on $\partial\mathbb{B}$, its retardation with respect to any $(X,t) \in \mathbb{R}^4$ is defined, as in [V.36.i], as a function on $\partial\mathbb{B}(X,t)$. In keeping with the notation already introduced, we shall use the symbol $[\chi^{-1}]_{[X,t]}$ for this retardation. Thus,

$$[\chi^{-1}]_{[X,t]}(Y) := \chi^{-1}_{t-\frac{1}{c}r_X(Y)}(Y), \qquad \text{for each} \qquad Y \in \partial\mathbb{B}(X,t).$$

We can identify this as the inverse of the homeomorphism $[\chi]_{(X,t)}$ of ∂R onto $\partial\mathbb{B}(X,t)$; this is the essential content of the next observation.

[V.40] P R O P O S I T I O N. *Let M be a motion, (R,χ) a reference pair for M, and τ the associated retardation function. Let* $X \in \mathbb{R}^3$ *and* $t \in \mathbb{R}$. *Then*

(i) $$[\chi]^{-1}_{(X,t)} = [\chi^{-1}]_{[X,t]}; \tag{1}$$

(ii) $$\tau(\chi^{-1}_{t-\frac{1}{c}r_X(Y)}(Y);X,t) = \frac{1}{c}r_X(Y),$$

i.e.

$$\tau([\chi]^{-1}_{(X,t)}(Y);X,t) = \frac{1}{c}\, r_X(Y), \qquad \text{for each} \qquad Y \in \partial\mathbb{B}(X,t). \tag{2}$$

P R O O F. Let $Y \in \partial\mathbb{B}(X,t)$. Then $P_Y := [\chi]^{-1}_{(X,t)}(Y)$ is the unique point of ∂R for which $Y = [\chi]_{(X,t)}(P_Y) = \chi(P_Y,\ t-\tau(P_Y;X,t))$, so $P_Y = \chi^{-1}_{t-\tau(P_Y;X,t)}(Y)$. Since $\tau(P_Y;X,t) = \frac{1}{c}\, r_X(\chi(P_Y,\ t-\tau(P_Y;X,t)))$, we find that $\tau(P_Y;X,t) = \frac{1}{c}\, r_X(Y)$, whence $P_Y = \chi^{-1}_{t-\frac{1}{c} r_X(Y)}(Y) = [\chi^{-1}]_{[X,t]}(Y)$. This gives the first statement of (ii), and also shows that $[\chi]^{-1}_{(X,t)}(Y) = [\chi^{-1}]_{[X,t]}(Y)$ for each $Y \in \partial\mathbb{B}(X,t)$, so $[\chi^{-1}]_{[X,t]}$ is an extension of $[\chi]^{-1}_{(X,t)}$. Since the domain of $[\chi^{-1}]_{[X,t]}$ is also $\partial\mathbb{B}(X,t)$, the equality in (i) follows. The second statement in (ii) is then an immediate consequence. $\square$.

The retardation facts also enable us to make an assertion concerning the intersection of any backward characteristic cone with $\partial\mathbb{B}$, for any motion M.

[V.41] P R O P O S I T I O N. *Let M be a motion, (R,χ) a reference pair for M, and τ the associated retardation function. Let $X \in \mathbb{R}^3$, $t \in \mathbb{R}$. Then the map $[\chi^*]_{(X,t)}$,*

$$[\chi^*]_{(X,t)}(P) = ([\chi]_{(X,t)}(P),\ t-\tau(P;X,t)), \qquad P \in \partial R,$$

provides a homeomorphism of ∂R onto $\partial\mathbb{B} \cap C_-(X,t)$.

P R O O F. It is a simple matter to check that $P_{(X,t)}|\ \partial\mathbb{B}(X,t)$ is a homeomorphism of $\partial\mathbb{B}(X,t)$ onto $\partial\mathbb{B}\cap C_-(X,t)$. In fact, $P_{(X,t)}$ is a continuous injection of $\mathbb{R}^3$ onto $C_-(X,t)$, so $P_{(X,t)}|\ \partial\mathbb{B}(X,t)$

is a continuous injection of $\partial\mathbb{B}(X,t)$ into $C_-(X,t)$. But, if $Z \in \partial\mathbb{B}(X,t)$, then $Z \in \partial B_{t-\frac{1}{c}r_X(Z)}$, so $P_{(X,t)}(Z) = (Z, t-\frac{1}{c}r_X(Z)) \in \cup_{\zeta\in\mathbb{R}}\{\partial B_\zeta\times\{\zeta\}\} = \partial\mathbb{B}$, so the function is into $\partial\mathbb{B}\cap C_-(X,t)$. If $(Z,\zeta) \in \partial\mathbb{B}\cap C_-(X,t)$, then $Z \in \partial B_\zeta$ and $\zeta = t-\frac{1}{c}r_X(Z)$, so $Z \in \partial B_{\zeta-\frac{1}{c}r_X(Z)}$, *i.e.*, $Z \in \partial\mathbb{B}(X,t)$. Since $P_{(X,t)}(Z) = (Z, t-\frac{1}{c}r_X(Z)) = (Z,\zeta)$, we can conclude that $P_{(X,t)}|\ \partial\mathbb{B}(X,t)$ is a continuous bijection of $\partial\mathbb{B}(X,t)$ onto $\partial\mathbb{B}\cap C_-(X,t)$. The inverse of this function is just $Pr|\ \partial\mathbb{B}\cap C_-(X,t)$, which is continuous.

Thus, $(P_{(X,t)}|\ \partial\mathbb{B}(X,t))\circ[\chi]_{(X,t)}: \partial R \to \partial\mathbb{B}\cap C_-(X,t)$ is a homeomorphism, in view of [V.38]. Since, for each $P \in \partial R$,

$$(P_{(X,t)}|\ \partial\mathbb{B}(X,t))\circ[\chi]_{(X,t)}(P) = ([\chi]_{(X,t)}(P),\ t-\frac{1}{c}r_X([\chi]_{(X,t)}(P)))$$

$$= ([\chi]_{(X,t)}(P),\ t-\tau(P;X,t)),$$

the proof is complete. □.

Note that, for any motion M, the sets ∂B_ζ $(\zeta \in \mathbb{R})$, $\partial\mathbb{B}(X,t)$, and $\partial\mathbb{B}\cap C_-(X,t)$ $((X,t) \in \mathbb{R}^4)$ are pairwise homeomorphic; in turn, each is homeomorphic to ∂R, where R is any reference set for M. Also, we have shown that $\partial\mathbb{B}$ is homeomorphic to any $\partial R\times\mathbb{R}$.

Under the assumption that a reference function possess a certain number of continuous derivatives with respect to its fourth argument, the corresponding retardation function possesses a corresponding number of continuous partial derivatives with respect to its second set of arguments. More precisely:

[V.42] P R O P O S I T I O N. *Let* M *be a motion,* (R,χ) *a reference pair for* M, *and* τ *the associated retardation function.*

(*i*) *Suppose, for some* $P \in \partial R$, $\chi(P,\cdot) \in C^k(\mathbb{R};\mathbb{R}^3)$, *where* $k \in \mathbb{N}\cup\{\infty\}$. *Then* $\tau(P;\cdot,\cdot) \in C^k(\mathbb{R}^4\cap\{(\chi(P,\zeta),\zeta) \mid \zeta \in \mathbb{R}\}')$. *Moreover, whenever* $(X,t) \in \mathbb{R}^4\cap\{(\chi(P,\zeta),\zeta) \mid \zeta \in \mathbb{R}\}'$, *i.e., whenever* $(X,t) \in \mathbb{R}^4$ *and* $X \neq \chi(P,t)$, *then*

$$\tau;_4(P;X,t) = \frac{r_{X,\ell}([\chi]_{(X,t)}(P))\cdot[\chi^{\ell^c}_{;4}]_{(X,t)}(P)}{1+r_{X,\ell}([\chi]_{(X,t)}(P))\cdot[\chi^{\ell^c}_{;4}]_{(X,t)}(P)}, \tag{1}$$

$$1-\tau;_4(P;X,t) = \frac{1}{1+r_{X,\ell}([\chi]_{(X,t)}(P))\cdot[\chi^{\ell^c}_{;4}]_{(X,t)}(P)}, \tag{2}$$

$$\tau;_4(P;X,t) = \{1-\tau;_4(P;X,\tilde{t})\}\cdot\{r_{X,\ell}([\chi]_{(X,t)}(P))\cdot[\chi^{\ell^c}_{;4}]_{(X,t)}(P)\}, \tag{3}$$

and

$$\tau;_i(P;X,t) = -\frac{1}{c}\{1-\tau;_4(P;X,t)\}\cdot r_{X,i}([\chi]_{(X,t)}(P)), \qquad i = 1,2,3. \tag{4}$$

If $k > 1$ $[k = \infty]$, *then each partial derivative of* $\tau(P;\cdot,\cdot)$ *of order* > 1 *and* $\leq k$ [*of any order*] *can be computed in* $\mathbb{R}^4\cap\chi^*(P,\mathbb{R})'$ *by successive differentiation of* (1) *and* (4).

(*ii*) *Suppose* $D_4^j\chi \in C(\partial R\times\mathbb{R};\mathbb{R}^3)$ *for* $j = 1,\ldots,k$, *for some* $k \in \mathbb{N}$ [*for each* $j \in \mathbb{N}$]. *If the degree of the multi-index* $\alpha = (\alpha_1,\ldots,\alpha_4)$ *is* $\leq k$ [*for any* α], *then* $\tau;_\alpha \mid \partial R\times\{\mathbb{B}^q\cup\Omega^\sigma\} \in C(\partial R\times\{\mathbb{B}^q\cup\Omega^\sigma\})$.

Before presenting the proof, we remark that (1) and (4) are simply the results which would be obtained by formal implicit differentiation of (V.34.1), implicitly defining τ. Indeed, a very short proof can be constructed by first invoking the implicit function theorem to show that τ possesses the requisite derivatives, then effecting the implicit differentiation of (V.34.1) (cf., the proof of [V.47.v]). However, we choose to follow a more instructive (albeit much longer) method of proof which provides practice in manipulating τ.

P R O O F. (i) $P \in \partial R$ is fixed, with $\chi(P,\cdot) \in C^k(\mathbb{R},\mathbb{R}^3)$. It is easy to see that $\{(\chi(P,\zeta),\zeta) \mid \zeta \in \mathbb{R}\}$ is a closed subset of $\mathbb{R}^4$, since the continuity of $\chi(P,\cdot)$ on $\mathbb{R}$ implies that the limit of any convergent sequence in the set must also be in the set. Thus, $\{(\chi(P,\zeta),\zeta) \mid \zeta \in \mathbb{R}\}'$ is open in $\mathbb{R}^4$. Choose $(X,t) \in \{(\chi(P,\zeta),\zeta) \mid \zeta \in \mathbb{R}\}'$. Then $X \neq \chi(P,t)$, so $\tau(P;X,t) > 0$, by [V.35.i]. Then $d := r_X(\chi(P, t-\tau(P;X,t))) = c\tau(P;X,t) > 0$, *i.e.*, $X \neq \chi(P, t-\tau(P;X,t))$. According to (V.35.1), we have

$$|\tau(P;X,t+h)-\tau(P;X,t)| < \frac{d}{2c} \quad \text{whenever} \quad |h| < \frac{c-c^*}{2cc^*}\,d. \tag{5}$$

Now, we claim that if $|h| < \min\left\{\frac{d}{2c}, \frac{c-c^*}{2cc^*}\,d\right\}$, then X does not lie on the closed line segment joining $\chi(P, t-\tau(P;X,t))$ and $\chi(P,t+h-\tau(P;X,t+h))$ (this line segment may be degenerate, consisting of the single point $\chi(P,t-\tau(P;X,t))$; in that case, we already know the claim to be true). To see this, assume the contrary: for some $h \in \mathbb{R}$ with $|h| < \min\left\{\frac{d}{2c}, \frac{c-c^*}{2cc^*}\,d\right\}$, there exists $\lambda \in [0,1]$ such that

$$X = \chi(P,t-\tau(P;X,t))+\lambda\{\chi(P,t+h-\tau(P;X,t+h))-\chi(P,t-\tau(P;X,t))\}.$$

Then

$$\begin{aligned} d &\le \lambda c^*|h-\{\tau(P;X,t+h)-\tau(P;X,t)\}| \\ &\le c^*\{|h|+|\tau(P;X,t+h)-\tau(P;X,t)|\} \\ &\le c^*\left\{\frac{d}{2c}+\frac{d}{2c}\right\} = \frac{c^*}{c}\,d < d. \end{aligned}$$

This impossibility proves the claim.

Suppose then that $0 < |h| < \min\left\{\frac{d}{2c}, \frac{c-c^*}{2cc^*}\,d\right\}$. Because of the fact just proven, we can apply the mean-value theorem to the function r_X in the following manner:

$$\frac{1}{h}\{\tau(P;X,t+h)-\tau(P;X,t)\}$$

$$\begin{aligned} &= \frac{1}{hc}\{r_X(\chi(P,t+h-\tau(P;X,t+h)))-r_X(\chi(P,t-\tau(P;X,t)))\} \\ &= \frac{1}{hc}\, r_{X,\ell}(Y_h)\cdot\{\chi^\ell(P,t+h-\tau(P;X,t+h))-\chi^\ell(P,t-\tau(P;X,t))\}, \end{aligned}$$

where Y_h is some point on the line segment joining the points $\chi(P,t-\tau(P;X,t))$ and $\chi(P,t+h-\tau(P;X,t+h))$. Again applying the mean-value theorem, this time to each of the coordinate functions $\chi^\ell \in C^1(\mathbb{R})$, $\ell = 1,2,3$, we can assert that there exist $t_h^\ell \in \mathbb{R}$, $\ell = 1,2,3$, in the open interval with endpoints $t-\tau(P;X,t)$ and $t+h-\tau(P;X,t+h)$ (which cannot be equal, for then we should have, from above, $\tau(P;X,t) = \tau(P;X,t+h)$, giving $h = 0$, contrary to our assumption) such that[+]

[+]We write $\chi^{\ell^c}_{,4} := \frac{1}{c}\chi^\ell_{,4}$; cf., [V.43.c], *infra*.

$$\frac{1}{h}\{\tau(P;X,t+h)-\tau(P;X,t)\}$$

$$=\frac{1}{h}\, r_{X,\ell}(Y_h)\cdot \chi^{\ell^c}_{,4}(P,t^\ell_h)\cdot\{h-\{\tau(P;X,t+h)-\tau(P;X,t)\}\};$$

we have abused the summation convention on the right-hand side here, but the meaning should be clear: sum on ℓ over $\{1,2,3\}$. A rearrangement of the latter equality produces

$$\{1+r_{X,\ell}(Y_h)\cdot\chi^{\ell^c}_{,4}(P,t^\ell_h)\}\cdot\frac{1}{h}\{\tau(P;X,t+h)-\tau(P;X,t)\}$$
$$= r_{X,\ell}(Y_h)\cdot\chi^{\ell^c}_{,4}(P,t^\ell_h), \tag{6}$$

whenever $0<|h|<\min\left\{\frac{d}{2c}, \frac{c-c^*}{2cc^*}d\right\}$. For no such h can we have $r_{X,\ell}(Y_h)\cdot\chi^{\ell^c}_{,4}(P,t^\ell_h)=-1$, by the very fact that equality (6) holds for each such h; thus, (6) determines the difference quotient appearing in its left-hand side, for each such h. Now,

$$|Y_h-\chi(P,t-\tau(P;X,t))|_3 \le |\chi(P,t+h-\tau(P;X,t+h))-\chi(P,t-\tau(P;X,t))|_3,$$

and

$$|t^\ell_h-\{t-\tau(P;X,t)\}| \le |h-\{\tau(P;X,t+h)-\tau(P;X,t)\}|$$
$$\le |h|+|\tau(P;X,t+h)-\tau(P;X,t)|, \qquad \ell=1,2,3,$$

for these same h, whence the continuity of $\tau(P;X,\cdot)$ and $\chi(P,\cdot)$ shows that $\lim_{h\to 0} Y_h=\chi(P,t-\tau(P;X,t))$, $\lim_{h\to 0} t^\ell_h = t-\tau(P;X,t)$, $\ell=1,2,3$. The continuity of $\chi_{,4}(P,\cdot)$ gives, in turn, $\lim_{h\to 0}\chi^\ell_{,4}(P,t^\ell_h)=\chi^\ell_{,4}(P,t-\tau(P;X,t))$, $\ell=1,2,3$. Since we showed that $Y_h\ne X$ for all h in question, and $X\ne\chi(P,t-\tau(P;X,t))$, we also see that $\lim_{h \to 0} r_{X,\ell}(Y_h) = r_{X,\ell}(\chi(P,t-\tau(P;X,t)))$, $\ell = 1,2,3$.

Finally, we show that $1+r_{X,\ell}(\chi(P,t-\tau(P;X,t)))\cdot\chi_{,4}^{\ell^c}(P,t-\tau(P;X,t)) \neq 0$: first, observe that (V.1.1) implies the validity of the inequality $|\chi_{,4}(P,\zeta)|_3 \leq c^*$, for each $\zeta \in \mathbb{R}$, while $|\text{grad } r_X(Y)|_3 = 1$ whenever $Y \neq X$. Then the Cauchy-Schwarz inequality yields

$$\begin{aligned} &1+r_{X,\ell}(\chi(P,t-\tau(P;X,t)))\cdot\chi_{,4}^{\ell^c}(P,t-\tau(P;X,t)) \\ &\geq 1-|r_{X,\ell}(\chi(P,t-\tau(P;X,t)))\cdot\chi_{,4}^{\ell^c}(P,t-\tau(P;X,t))| \qquad (7) \\ &\geq 1-\frac{c^*}{c} > 0. \end{aligned}$$

Upon computing the difference quotient $\frac{1}{h}\{\tau(P;X,t+h)-\tau(P;X,t)\}$ from (6) and employing these facts to allow $h \to 0$, it is found that $\tau_{;4}(P;X,t)$ exists and is given by the expression appearing in (1).

Obviously, (2) and (3) are immediate consequences of (1).

The proof of (4) is similar to that of (1). Choose $i \in \{1,2,3\}$. For $h \in \mathbb{R}$, let $h_i := he_i^{(3)}$. Again by (V.35.1),

$$|\tau(P;X+h_i,t)-\tau(P;X,t)| < \frac{d}{c} \quad \text{whenever} \quad |h| < \frac{c-c^*}{c}d,$$

and from this we deduce, as before, that if $|h| < \frac{c-c^*}{c}d$, then X does not lie on the (possibly degenerate) closed line segment joining $\chi(P,t-\tau(P;X+h_i,t))$ and $\chi(P,t-\tau(P;X,t))$. Next, observe that the function $(Y,Z) \mapsto Y-\chi(P,t-\tau(P;Z,t))$ is continuous on $\mathbb{R}^3\times\mathbb{R}^3$ and is non-zero at (X,X) (recall that $X \neq \chi(P,t-\tau(P;X,t))$). Hence, there is some $\delta > 0$ for which $|Y-\chi(P,t-\tau(P;Z,t))|_3 > 0$ whenever

$Y,Z \in B_\delta(X)$. Supposing that $|h| < \delta$, this statement is true when Y is any point on the closed line segment joining X and $X+h_i$, and $Z = X+h_i$. That is, if $|h| < \delta$, then $\chi(P,t-\tau(P;X+h_i,t))$ does not lie on the closed line segment joining X and $X+h_i$. We use these observations to justify two applications of the mean-value theorem to the distance function in the following computation; as before, we also apply the mean-value theorem to each $\chi^\ell(P,\cdot)$, $\ell = 1,2,3$: suppose that $0 < |h| < \min\left\{\frac{c-c^*}{c} d, \delta\right\}$. Then

$$\frac{1}{h}\{\tau(P;X+h_i,t)-\tau(P;X,t)\}$$

$$= \frac{1}{hc}\{r_{X+h_i}(\chi(P,t-\tau(P;X+h_i,t)))-r_X(\chi(P,t-\tau(P;X+h_i,t)))\}$$

$$+ \frac{1}{hc}\{r_X(\chi(P,t-\tau(P;X+h_i,t)))-r_X(\chi(P,t-\tau(P;X,t)))\}$$

$$= \frac{1}{c} r_{\chi(P,t-\tau(P;X+h_i,t)),i}(X_h)$$

$$- \frac{1}{hc} r_{X,\ell}(Z_h)\cdot\{\chi^\ell(P,t-\tau(P;X+h_i,t))-\chi^\ell(P,t-\tau(P;X,t))\}$$

$$= \frac{1}{c} r_{\chi(P,t-\tau(P;X+h_i,t)),i}(X_h)$$

$$- \frac{1}{h} r_{X,\ell}(Z_h)\cdot\chi^{\ell^c}_{,4}(P,\hat{t}^\ell_h)\cdot\{\tau(P;X+h_i,t)-\tau(P;X,t)\},$$

where X_h is some point on the line segment joining X and $X+h_i$, Z_h is on the line segment joining $\chi(P,t-\tau(P;X,t))$ and $\chi(P,t-\tau(P;X+h_i,t))$, and the $\hat{t}^\ell_h \in \mathbb{R}$, $\ell = 1,2,3$, are in the open interval with endpoints $t-\tau(P;X+h_i,t)$ and $t-\tau(P;X,t)$, if the latter are distinct; if these two numbers are equal, we choose, as we may, $\hat{t}^\ell_h = t-\tau(P;X,t)$. We find, for $0 < |h| < \min\left\{\frac{c-c^*}{c} d, \delta\right\}$,

$$\{1+r_{X,\ell}(Z_h)\cdot\chi^{\ell^c}_{;4}(P,\hat{t}^{\ell}_h)\}\ \frac{1}{h}\{\tau(P;X+h_i,t)-\tau(P;X,t)\}$$
$$= -\frac{1}{c}\ r_{X_h,i}(\chi(P,t-\tau(P;X+h_i,t))). \tag{8}$$

Reasoning as in the proof of (1), it is easy to show that $\lim_{h\to 0} X_h = X$, $\lim_{h\to 0} Z_h = \chi(P,t-\tau(P;X,t))$, $\lim_{h\to 0} \hat{t}^{\ell}_h = t-\tau(P;X,t)$, $\lim_{h\to 0} \chi^{\ell}_{;4}(P,\hat{t}^{\ell}_h) = \chi^{\ell}_{;4}(P,t-\tau(P;X,t))$, $\lim_{h\to 0} r_{X,\ell}(Z_h) =$ $r_{X,\ell}(\chi(P,t-\tau(P;X,t))$, for $\ell = 1,2,3$, and $\lim_{h\to 0} r_{X_h,i}(\chi(P,t-$ $\tau(P;X+h_i,t))) = r_{X,i}(\chi(P,t-\tau(P;X,t)))$; for the latter two results, we use the fact that $X \neq \chi(P,t-\tau(P;X,t))$. Then, recalling (7), we must note that

$$\lim_{h\to 0}\ \{1+r_{X,\ell}(Z_h)\cdot\chi^{\ell^c}_{;4}(P,\hat{t}^{\ell}_h)\} \geq 1-\frac{c^*}{c} > 0,$$

so there is some $\eta \in \left(0,\ \min\left\{\frac{c-c^*}{c}\ d,\ \delta\right\}\right)$ for which $1+r_{X,\ell}(Z_h)\cdot\chi^{\ell^c}_{;4}(P,\hat{t}^{\ell}_h) > \frac{1}{2}\left(1-\frac{c^*}{c}\right) > 0$ whenever $0 < |h| < \eta$. For these h, the difference quotient $\frac{1}{h}\{\tau(P;X+h_i,t)-\tau(P;X,t)\}$ can be computed from (8). Carrying this out and using the facts cited above, we can let $h \to 0$ and obtain

$$\tau_{;i}(P;X,t) = \frac{-\frac{1}{c}\ r_{X,i}(\chi(P,t-\tau(P;X,t)))}{1+r_{X,\ell}(\chi(P,t-\tau(P;X,t)))\cdot\chi^{\ell^c}_{;4}(P,t-\tau(P;X,t))}; \tag{9}$$

in view of (2), (9) is just (4).

We have shown that $\tau_{;i}(P;\cdot,\cdot)$, $i = 1,2,3$, and $\tau_{;4}(P;\cdot,\cdot)$ exist on $\{(\chi(P,\zeta),\zeta)\mid\ \zeta \in \mathbb{R}\}'$. From their explicit expressions appearing in (1) and (4) (or (9)), the continuity of these functions is obvious. Again from these expressions, it is equally obvious that, if $k \geq 2$, $\tau;_{ij}(P;\cdot,\cdot)$ and $\tau;_{4i}(P;\cdot,\cdot)$ exist on $\{(\chi(P,\zeta),\zeta) \mid \zeta \in \mathbb{R}\}'$ and are continuous there, whenever $i,j \in \{1,2,3,4\}$. Indeed the existence and continuity of higher-order partial derivatives of $\tau(P;\cdot,\cdot)$ on $\{(\chi(P,\zeta),\zeta) \mid \zeta \in \mathbb{R}\}'$ is limited only by the value of k, because of the chain rule, (1), (4), (8), and the fact that $\tau(P;\cdot,\cdot)$ is positive on $\{(\chi(P,\zeta)\zeta) \mid \zeta \in \mathbb{R}\}'$, so the function $(X,t) \mapsto r_X([\chi]_{(X,t)}(P))$ is also positive on this set. Consequently, the proof of (i) can be completed by induction.

(ii) Now, the hypotheses of (i) hold for some $k \in \mathbb{N} \cup \{\infty\}$ and each $P \in \partial\mathcal{R}$. Thus, each partial derivative of τ with respect to its second set of arguments, of order $\leq k$ if $k \in \mathbb{N}$ or any order if $k = \infty$, exists on $\{(P,X,t) \mid P \in \partial\mathcal{R},\ (X,t) \in \{(\chi(P,\zeta),\zeta) \mid \zeta \in \mathbb{R}\}'\} \subset \mathbb{R}^7$, and can be obtained from (1), (4), or differentiation thereof. In particular, each such partial derivative exists on $\partial\mathcal{R}\times\{\mathbb{B}^o \cup \Omega^\sigma\}$, for if $P \in \partial\mathcal{R}$ and $(X,t) \in \mathbb{B}^o \cup \Omega^\sigma$, then $X \in \mathcal{B}_t^o \cup \mathcal{B}_t'$, so $X \in (\partial\mathcal{B}_t)'$, so we must have $X \neq \chi(P,t)$, whence $(X,t) \in \{(\chi(P,\zeta),\zeta) \mid \zeta \in \mathbb{R}\}'$. The continuity of each such partial derivative on $\partial\mathcal{R}\times\{\mathbb{B}^o \cup \Omega^\sigma\}$ follows from the continuity hypotheses on $\{D_4^j\chi\}_{j=1}^k$ and from (1) and (4), either directly or by induction.

We shall allow these remarks to suffice for the completion of the proof. $\square$.

We shall next introduce, and derive the most important properties of, our fundamental classes of smooth motions, denoted by $\mathbb{M}(q)$ $(q \in \mathbb{N}\cup\{\infty\})$; even though it also proves convenient to consider motions possessing properties stronger than those imposed here, each family of smooth motions identified in [I.5] and Part IV turns out to be a subclass of some $\mathbb{M}(q)$, and so possesses the important characteristics which we are about to point out. Generally speaking, a motion $\mathcal{M}$ will be said to be in a certain smoothness class in $\mathbb{M}$ provided there is a reference pair $(\mathcal{R},\chi)$ for $\mathcal{M}$ such that $\partial\mathcal{R}$ and χ satisfy certain smoothness conditions.

[V.43] N O T A T I O N S. (a) Let M be an $(r,n;q)$-manifold. Recall that $M\times\mathbb{R}$ is then an $(r+1, n+1; q)$-manifold. Whenever (U,h) is a coordinate system in M, set

$$U^* := U\times\mathbb{R}$$

and define $h^*\colon\ U^* \to \mathbb{R}^{r+1}$ by

$$h^*(x,s) := (h(x),s), \quad \text{for each} \quad x \in U, \quad s \in \mathbb{R}. \tag{1}$$

Then it is easy to see that (U^*,h^*) is a coordinate system in $M\times\mathbb{R}$. Moreover,

$$h^*(U^*) = h(U)\times\mathbb{R}, \tag{2}$$

and

$$h^{*-1}(\hat{x},s) = (h^{-1}(\hat{x}),s), \quad \text{for each} \quad (\hat{x},s)\in h^*(U^*). \tag{3}$$

Observe that, if $\{(U_\iota,h_\iota)\}_{\iota\in I}$ is a covering collection of coordinate systems in M, then $\{(U_\iota^*, h_\iota^*)\}_{\iota\in I}$ is a covering collection of coordinate systems in $M\times\mathbb{R}$.

(b) Again with M an $(r,n;q)$-manifold, suppose that $f\colon M\times\mathbb{R} \to \mathbb{R}^m$ is a function such that $f_\zeta := f(\cdot,\zeta)\colon M \to \mathbb{R}^m$ is in $C^k(M;\mathbb{R}^m)$ for each $\zeta \in \mathbb{R}$ (for some $k \in \mathbb{N}$). Then $Jf_\zeta\colon M \to [0,\infty)$ is defined for each $\zeta \in \mathbb{R}$ (cf., Definition [I.2.10.iii]), and we can define $\mathfrak{J}f\colon M\times\mathbb{R} \to [0,\infty)$ according to

$$\mathfrak{J}f(x,\zeta) := Jf_\zeta(x), \qquad \text{for each} \quad x \in M, \quad \zeta \in \mathbb{R}. \tag{4}$$

(c) Whenever f is a function valued in either $\mathbb{K}$ or some $\mathbb{R}^m$, we shall write, as we have already in [V.42],

$$f^c := \frac{1}{c} f, \tag{5}$$

which should cause no conflicts with other notations.

(d) Let $\mathcal{M}$ be a motion, and $(\mathcal{R},\chi)$ a reference pair for $\mathcal{M}$. Recalling [V.15.vi], $\chi^*\colon \partial\mathcal{R}\times\mathbb{R} \to \partial\mathbb{B}$ is a homeomorphism. In particular, this allows us to associate with any function f on $\partial\mathbb{B}$ a function $\mathring{f}$ on $\partial\mathcal{R}\times\mathbb{R}$ *via*

$$\mathring{f} := f\circ\chi^*; \tag{6}$$

explicitly,

$$\mathring{f}(P,\zeta) := f\circ\chi^*(P,\zeta) = f(\chi(P,\zeta),\zeta) \qquad \text{for} \quad P \in \partial\mathcal{R}, \quad \zeta \in \mathbb{R}. \tag{7}$$

[V.44] R E M A R K S. Let $\mathcal{M}$ be a motion, $(\mathcal{R},\chi)$ a reference pair for M, and f a function on $\partial\mathbb{B}$. Obviously, f can be recovered from $\mathring{f}$: $f = \mathring{f}\circ\chi^{*-1}$, or, explicitly,

$$f(Z,\zeta) = \mathring{f}\circ\chi^{*-1}(Z,\zeta) = \mathring{f}(\chi_\zeta^{-1}(Z),\zeta) \quad \text{for each} \quad (Z,\zeta) \in \partial\mathbb{B}. \tag{1}$$

Choose $(X,t) \in \mathbb{R}^4$ and note that the two retardations $[f]_{[X,t]}$ (on $\partial\mathbb{B}(X,t)$) and $[\mathring{f}]_{(X,t)}$ (on ∂R) are defined (cf., [V.36]). We wish to point out here the relation

$$[\mathring{f}]_{(X,t)} = [f]_{[X,t]}\circ[\chi]_{(X,t)} \quad \text{on} \quad \partial R \tag{2}$$

between these functions. To see that (2) is correct, let τ denote, as always, the retardation function associated with R and χ; choosing $P \in \partial R$ and applying the appropriate definitions,

$$\begin{aligned}
[\mathring{f}]_{(X,t)}(P) &:= \mathring{f}(P,t-\tau(P;X,t)) \\
&:= f(\chi(P,t-\tau(P;X,t)),t-\tau(P;X,t)) \\
&= f(\chi(P,t-\tau(P;X,t)),t-\frac{1}{c}\,r_X(\chi(P,t-\tau(P;X,t)))) \\
&= [f]_{[X,t]}(\chi(P,t-\tau(P;X,t))) \\
&= [f]_{[X,t]}\circ[\chi]_{(X,t)}(P).
\end{aligned}$$

This implies (2).

[V.45] D E F I N I T I O N. Let M be a motion, and $q \in \mathbb{N}\cup\{\infty\}$. Then $M \in \mathbb{M}(q)$ iff M possesses a reference pair (R,χ) such that

(i) ∂R is a $(2,3;q)$-manifold,

(ii) $\chi \in C^q(\partial R\times\mathbb{R};\mathbb{R}^3)$,

and

(iii) rank $D\chi_\zeta(P) = 2$, for each $P \in \partial\mathcal{R}$, $\zeta \in \mathbb{R}$. ∎.

[V.46] R E M A R K S. Let M be a motion in some class $\mathbb{M}(q)$, and $(\mathcal{R},\chi)$ a reference pair for M as in [V.45].

(a) Clearly, $\partial\mathcal{R}\times\mathbb{R}$ is a $(3,4;q)$-manifold; recall that the inclusion [V.45.ii] means that, whenever (U,h) is a coordinate system in $\partial\mathcal{R}\times\mathbb{R}$, then $\chi\circ h^{-1} \in C^q(h(U);\mathbb{R}^3)$. In particular, for any coordinate system (U,h) in $\partial\mathcal{R}$, (U^*,h^*) is a coordinate system in $\partial\mathcal{R}\times\mathbb{R}$, so we have $\chi\circ h^{*-1} \in C^q(h^*(U^*),\mathbb{R}^3)$; explicitly, note that

$$\chi\circ h^{*-1}(\hat{x},s) = \chi(h^{-1}(\hat{x}),s) = \chi_s(h^{-1}(\hat{x})), \quad \text{for} \quad (\hat{x},s) \in h^*(U^*), \tag{1}$$

i.e., whenever $\hat{x} \in h(U)$ and $s \in \mathbb{R}$. It is, in fact, easy to see that $\chi \in C^q(\partial\mathcal{R}\times\mathbb{R};\mathbb{R}^3)$ iff there exists a *covering collection* $\{(U_\iota,h_\iota)\}_{\iota\in I}$ of coordinate systems in $\partial\mathcal{R}$ such that $\chi\circ h_\iota^{*-1} \in C^q(h_\iota^*(U_\iota^*);\mathbb{R}^3)$ for each $\iota \in I$: indeed, the sufficiency of this condition follows from [I.2.14] and the fact that $\{(U_\iota^*,h_\iota^*)\}_{\iota\in I}$ is a covering collection of coordinate systems in $\partial\mathcal{R}\times\mathbb{R}$, while the necessity is trivial.

(b) Let (U,h) be a coordinate system in $\partial\mathcal{R}$. Since $\chi\circ h^{*-1} \in C^q(h^*(U^*);\mathbb{R}^3)$, and $h^*(U^*) = h(U)\times\mathbb{R}$, equality (1) shows that $\chi_\zeta\circ h^{-1} \in C^q(h(U);\mathbb{R}^3)$ whenever $\zeta \in \mathbb{R}$. Thus, $\chi_\zeta \in C^q(\partial\mathcal{R};\mathbb{R}^3)$, for each $\zeta \in \mathbb{R}$, showing that [V.45.iii] makes sense, requiring that rank $D(\chi_\zeta\circ h^{-1})(h(P)) = 2$, or, equivalently, $J\chi_\zeta(P) > 0$, whenever $P \in U$ and $\zeta \in \mathbb{R}$.

(c) We can show that the derivatives $\{D_4^j \chi\}_{j=1}^q$ exist and are continuous on $\partial R\times\mathbb{R}$: let $P \in \partial R$, $\zeta \in \mathbb{R}$, and (U,h) be a coordinate system in ∂R, with $P \in U$. If $Q \in U$, $s \in \mathbb{R}$, and $\eta \in \mathbb{R}$, with $\eta \neq 0$, then

$$\frac{1}{\eta}\{\chi(Q,s+\eta)-\chi(Q,s)\} = \frac{1}{\eta}\{\chi(h^{-1}(h(Q)),s+\eta)-\chi(h^{-1}(h(Q)),s)\}$$

$$= \frac{1}{\eta}\{(\chi\circ h^{*-1})(h(Q),s+\eta)-(\chi\circ h^{*-1})(h(Q),s)\}.$$

Letting $\eta \to 0$, we find

$$D_4\chi(Q,s) = D_3(\chi\circ h^{*-1})(h(Q),s), \tag{2}$$

showing at once that $D_4\chi$ exists and is continuous on the neighborhood $U\times\mathbb{R}$ of (P,ζ). Since $\chi \in C^q(\partial R\times\mathbb{R};\mathbb{R}^3)$, the process can be repeated $q-1$ times, if $q \in \mathbb{N}$ with $q \geq 2$, or indefinitely, if $q = \infty$.

It is also important to note that

$$|D_4\chi(P,\zeta)|_3 \leq c^*, \qquad \text{for each} \qquad P \in \partial R, \qquad \zeta \in \mathbb{R}. \tag{3}$$

This bound is obtained directly from (V.1.1), since

$$\frac{1}{\eta}\{\chi(P,\zeta+\eta)-\chi(P,\zeta)\} \leq c^*,$$

if $P \in \partial R$, $\zeta \in \mathbb{R}$, and $\eta \in \mathbb{R}$, with $\eta \neq 0$.

(d) Since $\chi_\zeta \in C^q(\partial R;\mathbb{R}^3)$ for each $\zeta \in \mathbb{R}$, the function $\hat{J}\chi$ is defined on $\partial R\times\mathbb{R}$, as in [V.43.b]:

$$\hat{J}\chi(P,\zeta) := J\chi_\zeta(P), \qquad \text{for each} \qquad P \in \partial R, \qquad \zeta \in \mathbb{R}. \tag{4}$$

If $P \in \mathbb{R}$, $\zeta \in \mathbb{R}$, and $\{T_1(P), T_2(P)\}$ is any basis for $T_{\partial R}(P)$, then, by Definition [I.2.10.iii],

$$\hat{J}\chi(P,\zeta) = \frac{|D\chi_\zeta(P)T_1(P) \wedge D\chi_\zeta(P)T_2(P)|}{|T_1(P) \wedge T_2(P)|}.$$

However, if $\beta_i \in \mathbb{R}^3$ for $i = 1,2$, then $\beta_1 \times \beta_2$ is defined, and $|\beta_1 \wedge \beta_2| = |\beta_1 \times \beta_2|_3$ (cf. Fleming [15]). In the present case, we can therefore also write

$$\hat{J}\chi(P,\zeta) = \frac{|D\chi_\zeta(P)T_1(P) \times D\chi_\zeta(P)T_2(P)|_3}{|T_1(P) \times T_2(P)|_3}. \tag{5}$$

Now, if (U,h) is any coordinate system in ∂R, with $P \in U$, the basis $\{(h^{-1})_{,i}(h(P))\}_{i=1}^{2}$ for $T_{\partial R}(P)$ can be chosen, leading to the expression

$$\hat{J}\chi(P,\zeta) = \frac{|(\chi_\zeta \circ h^{-1})_{,1}(h(P)) \times (\chi_\zeta \circ h^{-1})_{,2}(h(P))|_3}{|(h^{-1})_{,1}(h(P)) \times (h^{-1})_{,2}(h(P))|_3}, \tag{6}$$

as in the derivation of (I.2.11.3). When we observe, from (1), that $(\chi_\zeta \circ h^{-1})_{,i}(h(P)) = (\chi \circ h^{*-1})_{,i}(h(P),\zeta)$, for $i = 1$ or 2, the local representation (6) implies that $\hat{J}\chi \in C(\partial R \times \mathbb{R})$, while if $q > 1$, we even have $\hat{J}\chi \in C^{q-1}(\partial R \times \mathbb{R})$; the details of the reasoning required to verify these inclusions are plain enough.

Finally, from [V.45.iii], it is clear that

$$\hat{J}\chi > 0 \quad \text{on} \quad \partial R \times \mathbb{R}. \tag{7}$$

The properties of the classes $\mathbb{M}(q)$, $q \in \mathbb{N} \cup \{\infty\}$, of which the principal ones are brought forth in the following statement, provide much of the basis for the reasoning employed in Parts I-IV.

[V.47] T H E O R E M. *Suppose* M *is a motion in* $\mathbb{M}(q)$, *for some* $q \in \mathbb{N}\cup\{\infty\}$. *Let* (R,χ) *be a reference pair for* M, *with the properties of Definition* [V.45].

(*i*) *Select any* $\zeta \in \mathbb{R}$:

(*1*) χ_ζ: $\partial R \to \mathbb{R}^3$ *is a* q-*imbedding, taking* ∂R *onto* ∂B_ζ;

(*2*) B_ζ^o *and* B_ζ' *are* q-*regular domains in* $\mathbb{R}^3$;

(*3*) *for each* $P \in \partial R$, $D\chi_\zeta(P)$: $T_{\partial R}(P) \to T_{\partial B_\zeta}(\chi_\zeta(P))$ *is a bijection.*

(*ii*) *Define* ν: $\partial\mathbb{B} \to \mathbb{R}^3$ *by*†

$$\nu(Z,\zeta) := \nu_{\partial B_\zeta}(Z), \qquad \textit{for each} \quad \zeta \in \mathbb{R}, \quad Z \in \partial B_\zeta, \tag{1}$$

and υ: $\partial\mathbb{B} \to \mathbb{R}$ *by*

$$\upsilon(Z,\zeta) := \nu(Z,\zeta)\bullet\chi_{,4}(\chi_\zeta^{-1}(Z),\zeta), \qquad \textit{for each} \quad \zeta \in \mathbb{R}, \quad Z \in \partial B_\zeta. \tag{2}$$

Then υ *is an intrinsic property of* M, *called its* **normal velocity**.

(*iii*) *Recall the definition of the function* χ^*, *set down in* [V.3]; *cf., also,* [V.15.vi].

(*1*) χ^*: $\partial R\times\mathbb{R} \to \mathbb{R}^4$ *is a* q-*imbedding, taking* $\partial R\times\mathbb{R}$ *onto* $\partial\mathbb{B}$; *in particular,* $J\chi^* > 0$ *on* $\partial R\times\mathbb{R}$;

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† We write $\nu_{\partial B_\zeta} := \nu_{\partial\{B_\zeta^o\}}$, for brevity.

(2) $\mathbb{B}^{o}$ *and* Ω^{σ} *are* q-*regular domains in* $\mathbb{R}^4$;

(3) *for each* $(P,\zeta) \in \partial R\times\mathbb{R}$, $D\chi^*(P,\zeta)$: $T_{\partial R\times\mathbb{R}}(P,\zeta) \to T_{\partial\mathbb{B}}(\chi^*(P,\zeta))$ *is a bijection;*

(4) *the exterior unit normal field* $\nu_{\partial\mathbb{B}}$: $\partial\mathbb{B} \to \mathbb{R}^4$ *for* $\partial\{\mathbb{B}^{o}\}$ *is given by*[†]

$$\nu_{\partial\mathbb{B}}(Z,\zeta) = \frac{1}{\sqrt{\{1+\upsilon^2(Z,\zeta)\}}}\{\nu^1(Z,\zeta)e_1^{(4)} - \upsilon(Z,\zeta)e_4^{(4)}\}, \tag{3}$$

for each $(Z,\zeta) \in \partial\mathbb{B}$;

(5) $J\chi^* \in C(\partial R\times\mathbb{R})$; $J\chi^*$ *is given by*

$$\begin{aligned} J\chi^*(P,\zeta) &= \sqrt{\{1+\upsilon^2(\chi(P,\zeta),\zeta)\}}\hat{J}\chi(P,\zeta) \\ &= \sqrt{\{1+\mathring{\upsilon}^2(P,\zeta)\}}\hat{J}\chi(P,\zeta), \quad \text{for} \quad (P,\zeta) \in \partial R\times\mathbb{R}. \end{aligned} \tag{4}$$

(iv) *The functions* ν, $\nu_{\partial\mathbb{B}}$, *and* υ *are continuous on* $\partial\mathbb{B}$. *If* $q \in \mathbb{N}$ *and* $q \geq 2$, *then* $\nu \in C^{q-1}(\partial\mathbb{B};\mathbb{R}^3)$, $\nu_{\partial\mathbb{B}} \in C^{q-1}(\partial\mathbb{B};\mathbb{R}^4)$, *and* $\upsilon \in C^{q-1}(\partial\mathbb{B})$, *while* $D_4^k\mathring{\nu}$, $D_4^k\hat{J}\chi$, *and* $D_4^k\mathring{\upsilon}$ *exist and are continuous on* $\partial R\times\mathbb{R}$ *for* $k = 1,\ldots,q-1$. *If* $q = \infty$, *the latter statement holds with the obvious modifications.*

(v) *Let* τ *be the retardation function associated with* (R,χ). *Whenever* $(X,t) \in \mathbb{B}^{o}\cup\Omega^{\sigma}$, *then* $\tau(\cdot;X,t) \in C^q(\partial R)$, *with*

[†] We write $\nu_{\partial\mathbb{B}} := \nu_{\partial\{\mathbb{B}^{o}\}}$, for brevity.

$$\tau(h^{-1}(\cdot);X,t)_{,i}(\hat{x})$$

$$= \frac{\frac{1}{c}\, r_{X,k}([\chi]_{(X,t)} \circ h^{-1}(\hat{x}))\cdot(\chi^{k}_{t-\tau(h^{-1}(\hat{x});X,t)} \circ h^{-1})_{,k}(\hat{x})}{1+r_{X,j}([\chi]_{(X,t)} \circ h^{-1}(\hat{x}))\cdot[\chi^{j}_{,4}]_{(X,t)} \circ h^{-1}(\hat{x})}, \tag{5}$$

for each $\hat{x} \in h(U)$, *and* $i = 1,2$,

where (U,h) *is any coordinate system in* ∂R.

(*vi*) *Fix* $(X,t) \in \mathbb{B}^{o} \cup \Omega^{\sigma}$:

(1) $[\chi]_{(X,t)}: \partial R \to \mathbb{R}^3$ *is a* q-*imbedding, taking* ∂R *onto* $\partial\mathbb{B}(X,t)$;

(2) $\mathbb{B}(X,t)^{o}$ *and* $\Omega^{\sigma}(X,t)$ *are* q-*regular domains in* $\mathbb{R}^3$;

(3) *for each* $P \in \partial R$, $D[\chi]_{(X,t)}(P)$: $T_{\partial R}(P) \to T_{\partial\mathbb{B}(X,t)}([\chi]_{(X,t)}(P))$ *is a bijection.*

(4) *Let* $P \in \partial R$. *An exterior normal for* $\partial\{\mathbb{B}(X,t)^{o}\}$ *at* $[\chi]_{(X,t)}(P)$ *is given by*

$$[\overset{o}{\nu}]_{(X,t)}(P)+[\overset{o}{\nu}{}^{c}]_{(X,t)}(P)\cdot \operatorname{grad} r_X([\chi]_{(X,t)}(P)). \tag{6}$$

(5) $J[\chi]_{(X,t)}$ *is given on* ∂R *by*

$$J[\chi]_{(X,t)}(P) = [\hat{J}\chi]_{(X,t)}(P)\cdot\{1-\tau_{;4}(P;X,t)\} \cdot\left|[\overset{o}{\nu}]_{(X,t)}(P)+[\overset{o}{\nu}{}^{c}]_{(X,t)}(P)\cdot \operatorname{grad} r_X([\chi]_{(X,t)}(P))\right|_3, \tag{7}$$

for each $P \in \partial R$.

P R O O F. (i) We have already remarked, in [V.2.g], that χ_ζ is a homeomorphism of ∂R onto ∂B_ζ, and, in [V.46.b], that $\chi_\zeta \in C^q(\partial R;\mathbb{R}^3)$. It is given, in [V.45.iii], that rank $D\chi_\zeta(P) = 2$, for each $P \in \partial R$. Thus, χ_ζ is a q-imbedding, and (i.1) is proven. In view of [I.2.17.i], $\partial B_\zeta = \chi_\zeta(\partial R)$ is a (2,3;q)-manifold. Now, [V.1.i] says that B_ζ is regularly closed, so B_ζ^o is regularly open (since $B_\zeta^o = (B_\zeta^{o-})^o$), and $B_\zeta^{o-\prime} = B_\zeta'$; referring to [I.2.29], we conclude that B_ζ^o and B_ζ' are q-regular domains in $\mathbb{R}^3$, since $\partial\{B_\zeta^o\} = \partial B_\zeta$ in this case. This proves (i.2). (i.3) follows immediately from [I.2.17.ii].

(ii) By (i) and [I.2.31], the unit exterior normal field $\nu_{\partial B_\zeta} := \nu_{\partial\{B_\zeta^o\}}$ for $\partial\{B_\zeta^o\}$ $(= \partial B_\zeta)$ exists for each $\zeta \in \mathbb{R}$, so ν and υ are well-defined by (1) and (2), respectively. To see that υ is intrinsic,[†] let $(\tilde{R},\tilde{\chi})$ be a reference pair for M. By definition, there exists a continuous bijection $F\colon \partial\tilde{R} \to \partial R$ such that $\tilde{\chi}(Q,s) = \chi(F(Q),s)$ for each $Q \in \partial\tilde{R}$, $s \in \mathbb{R}$. This shows at once that $D_4\tilde{\chi}$ exists, is in $C(\partial\tilde{R}\times\mathbb{R};\mathbb{R}^3)$, and

$$D_4\tilde{\chi}(Q,s) = D_4\chi(F(Q),s), \qquad \text{for} \qquad Q \in \partial\tilde{R}, \qquad s \in \mathbb{R}. \tag{8}$$

Now choose $\zeta \in \mathbb{R}$, then $Z \in \partial B_\zeta$. It is simple to show that $\chi_\zeta^{-1}(Z) = F(\tilde{\chi}_\zeta^{-1}(Z))$, which, with (8), yields

$$D_4\chi(\chi_\zeta^{-1}(Z),\zeta) = D_4\chi(F(\tilde{\chi}_\zeta^{-1}(Z)),\zeta) = D_4\tilde{\chi}(\tilde{\chi}_\zeta^{-1}(Z),\zeta),$$

so also

[†] Obviously, ν is intrinsic.

$$\nu(Z,\zeta)\bullet D_4\chi(\chi_\zeta^{-1}(Z),\zeta) = \nu(Z,\zeta)\bullet D_4\tilde{\chi}(\tilde{\chi}_\zeta^{-1}(Z),\zeta).$$

Thus, ν is intrinsic to M.

(iii) In [V.15.vi], we saw that χ^* provides a homeomorphism of $\partial R\times\mathbb{R}$ onto $\partial\mathbb{B}$, and we are given that $\chi^* \in C^q(\partial R\times\mathbb{R};\mathbb{R}^4)$. Consequently, to show that $\chi^*\colon \partial R\times\mathbb{R} \to \mathbb{R}^4$ is a q-imbedding, we must demonstrate that $\operatorname{rank} D\chi^*(P,\zeta) = 3$ for each $P \in \partial R$, $\zeta \in \mathbb{R}$, for which it suffices to show that $\operatorname{rank} D(\chi^* o h^{*-1})(\hat{x},\zeta) = 3$ for each $(\hat{x},\zeta) \in h^*(U^*) = h(U)\times\mathbb{R}$, whenever (U,h) is a coordinate system in ∂R. Choosing such a coordinate system, since $\chi^*(P,\zeta) := (\chi(P,\zeta),\zeta)$ for $P \in \partial R$, $\zeta \in \mathbb{R}$, we see that

$$\begin{aligned}\chi^* o h^{*-1}(\hat{y},s) &= (\chi o h^{*-1}(\hat{y},s),s) = (\chi(h^{-1}(\hat{y}),s),s)\\ &= (\chi_s o h^{-1}(\hat{y}),s), \quad \text{whenever} \quad \hat{y} \in h(U), \quad s \in \mathbb{R}.\end{aligned} \tag{9}$$

Then, since $\chi o h^{*-1} \in C^q(h^*(U^*);\mathbb{R}^3)$, and in view of (V.46.2), for $(\hat{x},\zeta) \in h^*(U^*)$, the matrix $((\chi^* o h^{*-1})^i_{,j}(\hat{x},\zeta))_{1\le i\le 4,\ 1\le j\le 3}$ of $D(\chi^* o h^{*-1})(\hat{x},\zeta)\colon \mathbb{R}^3 \to \mathbb{R}^4$ with respect to the standard basis vectors of $\mathbb{R}^3$ and $\mathbb{R}^4$ can clearly be written

$$\begin{pmatrix} (\chi_\zeta o h^{-1})^1_{,1}(\hat{x}) & (\chi_\zeta o h^{-1})^1_{,2}(\hat{x}) & \chi^1_{,4}(h^{-1}(\hat{x}),\zeta)\\ (\chi_\zeta o h^{-1})^2_{,1}(\hat{x}) & (\chi_\zeta o h^{-1})^2_{,2}(\hat{x}) & \chi^2_{,4}(h^{-1}(\hat{x}),\zeta)\\ (\chi_\zeta o h^{-1})^3_{,1}(\hat{x}) & (\chi_\zeta o h^{-1})^3_{,2}(\hat{x}) & \chi^3_{,4}(h^{-1}(\hat{x}),\zeta)\\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

We know that $\operatorname{rank} D(\chi_\zeta o h^{-1})(\hat{x}) = \operatorname{rank} D\chi_\zeta(h^{-1}(\hat{x})) = 2$, by condition

[V.45.iii], so the indicated upper left-hand submatrix in (10) has rank 2, whence it follows easily that the matrix (10) itself must have rank 3. Thus, χ^* is a q-imbedding of $\partial\mathcal{R}\times\mathbb{R}$ into $\mathbb{R}^4$, and (iii.1) is correct.

To prove (iii.2), we can now reason as in the proof of (i): directly from (iii.1) and [I.2.17.i], $\partial\mathbb{B} = \chi^*(\partial\mathcal{R}\times\mathbb{R})$ is a (3,4;q)-manifold. By [V.15.iv], $\mathbb{B}$ is regularly closed, so $\partial\{\mathbb{B}^o\} = \partial\mathbb{B}$, $\Omega^\sigma := \mathbb{B}' = \mathbb{B}^{o-'}$, and $\mathbb{B}^o$ is regularly open. Thus, [I.2.29] allows us to assert that $\mathbb{B}^o$ and Ω^σ are q-regular domains in $\mathbb{R}^4$.

(iii.3) is statement [I.2.17.ii], phrased in the present setting.

Since $\mathbb{B}^o$ is a q-regular domain, the exterior unit normal field $\nu_{\partial\mathbb{B}} := \nu_{\partial\{\mathbb{B}^o\}}$ for $\partial\{\mathbb{B}^o\}$ $(= \partial\mathbb{B})$ exists; we must show that the representation (3) holds. To see this, we begin by observing that if $Q \in \partial\mathcal{R}$, $s \in \mathbb{R}$, and $\{T_1(Q),T_2(Q)\}$ is a basis for $T_{\partial\mathcal{R}}(Q)$, then, by Remark [I.2.5], $\{T_1^i(Q)e_i^{(4)},\ T_2^i(Q)e_i^{(4)},\ e_4^{(4)}\}$ is a basis for $T_{\partial\mathcal{R}\times\mathbb{R}}(Q,s)$; by (iii.3), $D\chi^*(Q,s)$ carries the latter basis to a basis for $T_{\partial\mathbb{B}}(\chi^*(Q,s))$. Choose $\zeta \in \mathbb{R}$, $Z \in \partial\mathcal{B}_\zeta$ (so $(Z,\zeta) \in \partial\mathbb{B}$) and set $P := \chi_\zeta^{-1}(Z)$, so $(P,\zeta) = \chi^{*-1}(Z,\zeta)$. Let (U,h) be any coordinate system in $\partial\mathcal{R}$ with $P \in U$. Now, $Dh^{-1}(h(P)) \colon \mathbb{R}^2 \to T_{\partial\mathcal{R}}(P)$ is a linear bijection, so it takes any basis for $\mathbb{R}^2$ onto a basis for $T_{\partial\mathcal{R}}(P)$, which, in turn, $D\chi_\zeta(P)$ maps to a basis for $T_{\partial\mathcal{B}_\zeta}(Z)$, because of (i.3). Thus, we can construct the basis $\{T_{h1}(P),T_{h2}(P)\}$ for $T_{\partial\mathcal{R}}(P)$, where

$$T_{hi}(P) := Dh^{-1}(h(P))e_i^{(2)}, \quad i = 1,2, \tag{11}$$

thence producing the basis

$$\{D\chi_\zeta(P)T_{h1}(P),\ D\chi_\zeta(P)T_{h2}(P)\} \tag{12}$$

for $T_{\partial\mathcal{B}_\zeta}(Z)$ and the basis

$$\{D\chi^*(P,\zeta)T^i_{h1}(P)e_i^{(4)},\ D\chi^*(P,\zeta)T^i_{h2}(P)e_i^{(4)},\ D\chi^*(P,\zeta)e_4^{(4)}\} \tag{13}$$

for $T_{\partial\mathbb{B}}(Z,\zeta)$ (noting that $(Z,\zeta) = \chi^*(P,\zeta)$). We claim that the elements of the latter basis are given by

$$D\chi^*(P,\zeta)T^j_{hi}(P)e_j^{(4)} = (\chi_\zeta \circ h^{-1})^j_{,i}(h(P))e_j^{(4)}, \quad i = 1,2, \tag{14}$$

$$D\chi^*(P,\zeta)e_4^{(4)} = \chi^j_{,4}(P,\zeta)e_j^{(4)} + e_4^{(4)}. \tag{15}$$

To prove (14), observe first that the matrix of $Dh^{*-1}(h^*(P,\zeta)): \mathbb{R}^3 \to \mathbb{R}^4$ with respect to the standard basis vectors of $\mathbb{R}^3$ and $\mathbb{R}^4$ is, from (V.43.3),

$$((h^{*-1})^i_{,j}(h^*(P,\zeta)))_{\substack{1\le i\le 4\\ 1\le j\le 3}} = \begin{pmatrix} (h^{-1})^1_{,1}(h(P)) & (h^{-1})^1_{,2}(h(P)) & 0 \\ (h^{-1})^2_{,1}(h(P)) & (h^{-1})^2_{,2}(h(P)) & 0 \\ (h^{-1})^3_{,1}(h(P)) & (h^{-1})^3_{,2}(h(P)) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Thus, for $i = 1,2$,

$$\begin{aligned} Dh^{*-1}(h^*(P,\zeta))e_i^{(3)} &= (h^{-1})^j_{,i}(h(P))e_j^{(4)} = \{Dh^{-1}(h(P))e_i^{(2)}\}^j e_j^{(4)} \\ &= T^j_{hi}(P)e_j^{(4)}. \end{aligned} \tag{17}$$

Also, using the matrix (10), with $\hat{x} = h(P)$,

$$D(\chi^* o h^{*-1})(h^*(P,\zeta))e_i^{(3)} = (\chi_\zeta o h^{-1})^j_{,i}(h(P))e_j^{(4)}, \quad \text{for} \quad i = 1,2. \tag{18}$$

From the definition (I.2.10.1), and accounting for (17) and (18), we compute

$$\begin{aligned} D\chi^*(P,\zeta)T^j_{hi}(P)e_j^{(4)} &= D(\chi^* o h^{*-1})(h^*(P,\zeta)) o \{Dh^{*-1}(h^*(P,\zeta))\}^{-1} T^j_{hi}(P)e_j^{(4)} \\ &= D(\chi^* o h^{*-1})(h^*(P,\zeta))e_i^{(3)} \\ &= (\chi_\zeta o h^{-1})^j_{,i}(h(P))e_j^{(4)}, \quad \text{for} \quad i = 1,2, \end{aligned}$$

which is just (14). Proceeding to the verification of (15), we see that

$$Dh^{*-1}(h^*(P,\zeta))e_3^{(3)} = e_4^{(4)}, \tag{19}$$

from (16), while, from the matrix (10),

$$D(\chi^* o h^{*-1})(h^*(P,\zeta))e_3^{(3)} = \chi^j_{,4}(P,\zeta)e_j^{(4)} + e_4^{(4)}. \tag{20}$$

Using first (19), then (20),

$$\begin{aligned} D\chi^*(P,\zeta)e_4^{(4)} &= D(\chi^* o h^{*-1})(h^*(P,\zeta)) o \{Dh^{*-1}(h^*(P,\zeta))\}^{-1} e_4^{(4)} \\ &= D(\chi^* o h^{*-1})(h^*(P,\zeta))e_3^{(3)} = \chi^j_{,4}(P,\zeta)e_j^{(4)} + e_4^{(4)}, \end{aligned}$$

i.e., (15) holds. Let us use (14) and (15) to compute the scalar product of $\nu^k(Z,\zeta)e_k^{(4)} - \upsilon(Z,\zeta)e_4^{(4)}$ with each element of the basis for $T_{\partial B}(Z,\zeta)$ given by (13): for $i = 1,2$,

$$\{\nu^k(Z,\zeta)e_k^{(4)}-\upsilon(Z,\zeta)e_4^{(4)}\}\bullet(\chi_\zeta\circ h^{-1})^j_{,i}(h(P))e_j^{(4)}$$

$$=\nu^j(Z,\zeta)\cdot(\chi_\zeta\circ h^{-1})^j_{,i}(h(P))=\nu_{\partial\mathcal{B}_\zeta}(Z)\bullet D\chi_\zeta(P)T_{hi}(P)=0$$

(since $D\chi_\zeta(P)T_{hi}(P)\in T_{\partial\mathcal{B}_\zeta}(Z)$), and, recalling that $P=\chi_\zeta^{-1}(Z)$,

$$\{\nu^k(Z,\zeta)e_k^{(4)}-\upsilon(Z,\zeta)e_4^{(4)}\}\bullet\{\chi^j_{,4}(P,\zeta)e_j^{(4)}+e_4^{(4)}\}$$

$$=\nu(Z,\zeta)\bullet\chi_{,4}(\chi_\zeta^{-1}(Z),\zeta)-\upsilon(Z,\zeta)=0,$$

the latter equality by (2). Thus, $\nu^k(Z,\zeta)e_k^{(4)}-\upsilon(Z,\zeta)e_4^{(4)}\in N_{\partial\mathbb{B}}(Z,\zeta)$, and tne vector on the right-hand side of (3) is then a unit normal to $\partial\mathbb{B}$ at (Z,ζ). The proof of (iii.4) shall be complete once we have shown $\nu^k(Z,\zeta)e_k^{(4)}-\upsilon(Z,\zeta)e_4^{(4)}$ to be an *exterior* normal for $\partial\mathbb{B}$ at (Z,ζ). To secure this conclusion, we again use the fact that $\mathbb{B}^o$ is a q-regular domain, proven in (iii.2). Thus, there exist an open neighborhood $U_{(Z,\zeta)}$ of (Z,ζ) in $\mathbb{R}^4$ and a function $\Phi_{(Z,\zeta)}\in C^q(U_{(Z,\zeta)})$ such that grad $\Phi_{(Z,\zeta)}(Y,s)\neq 0$ for each $(Y,s)\in U_{(Z,\zeta)}$, $\partial\mathbb{B}\cap U_{(Z,\zeta)}=\partial\{\mathbb{B}^o\}\cap U_{(Z,\zeta)}=\{(Y,s)\in U_{(Z,\zeta)}\mid \Phi_{(Z,\zeta)}(Y,s)=0\}$, and $\mathbb{B}^o\cap U_{(Z,\zeta)}=\{(Y,s)\in U_{(Z,\zeta)}\mid \Phi_{(Z,\zeta)}(Y,s)<0\}$. By appealing to Remark [I.2.32.b], we find that grad $\Phi_{(Z,\zeta)}(Z,\zeta)$ is an exterior normal to $\partial\{\mathbb{B}^o\}$ $(=\partial\mathbb{B})$ at (Z,ζ). Since $N_{\partial\mathbb{B}}(Z,\zeta)$ is one-dimensional, there exists some $\alpha\in\mathbb{R}$ for which

$$\nu^k(Z,\zeta)e_k^{(4)}-\upsilon(Z,\zeta)e_4^{(4)}=\alpha\ \mathrm{grad}\ \Phi_{(Z,\zeta)}(Z,\zeta);\tag{21}$$

obviously, $\alpha\neq 0$. From (21), we can conclude that $\nu^k(Z,\zeta)e_k^{(4)}-\upsilon(Z,\zeta)e_4^{(4)}$ is an exterior normal for $\partial\{\mathbb{B}^o\}$ at (Z,ζ) by first showing that $\alpha>0$, which we now proceed to do. Since $\nu^\ell(Z,\zeta)\neq 0$ for some $\ell \in \{1,2,3\}$, (21) implies that $D_\ell \Phi_{(Z,\zeta)}(Z,\zeta) \neq 0$. Then we can find an $\varepsilon > 0$ such that $B^4_\varepsilon(Z,\zeta) \subset U_{(Z,\zeta)}$ and

$$D_\ell \Phi_{(Z,\zeta)}(Y,s) \neq 0 \quad \text{for each} \quad (Y,s) \in B^4_\varepsilon(Z,\zeta). \tag{22}$$

Note that

$$\text{grad } \Phi_{(Z,\zeta)}(Y,s) \neq 0 \quad \text{for each} \quad (Y,s) \in B^4_\varepsilon(Z,\zeta), \tag{23}$$

$$\partial\mathbb{B} \cap B^4_\varepsilon(Z,\zeta) = \{(Y,s) \in B^4_\varepsilon(Z,\zeta) \mid \Phi_{(Z,\zeta)}(Y,s) = 0\}, \tag{24}$$

and

$$\mathbb{B}^o \cap B^4_\varepsilon(Z,\zeta) = \{(Y,s) \in B^4_\varepsilon(Z,\zeta) \mid \Phi_{(Z,\zeta)}(Y,s) < 0\}. \tag{25}$$

Define $\Phi_Z\colon B^3_\varepsilon(Z) \to \mathbb{R}$ by setting

$$\Phi_Z(Y) := \Phi_{(Z,\zeta)}(Y,\zeta), \quad \text{for each} \quad Y \in B^3_\varepsilon(Z). \tag{26}$$

Clearly, $\Phi_Z \in C^q(B^3_\varepsilon(Z))$, and

$$D_i\Phi_Z(Y) = D_i\Phi_{(Z,\zeta)}(Y,\zeta), \quad \text{for each} \quad Y \in B^3_\varepsilon(Z) \text{ and } i \in \{1,2,3\}. \tag{27}$$

From (27) and (22), it follows readily that $\text{grad } \Phi_Z(Y) \neq 0$ for each $Y \in B^3_\varepsilon(Z)$. Further, it is a routine matter to prove that $\partial\mathcal{B}_\zeta \cap B^3_\varepsilon(Z) = \{Y \in B^3_\varepsilon(Z) \mid \Phi_Z(Y) = 0\}$, using (24), (26), and the fact that $\partial\mathbb{B} = \cup_{s\in\mathbb{R}} \{\partial\mathcal{B}_s \times \{s\}\}$, by [V.15.iii]. Similarly, the equality $\mathcal{B}^o_\zeta \cap B^3_\varepsilon(Z) = \{Y \in B^3_\varepsilon(Z) \mid \Phi_Z(Y) < 0\}$ is an easy consequence of (25), (26), and the representation $\mathbb{B}^o = \cup_{s\in\mathbb{R}} \{\mathcal{B}^o_s \times \{s\}\}$, which is from [V.15.ii]. Thus, $\text{grad } \Phi_Z(Z)$ is an exterior normal for $\partial\{\mathcal{B}^o_\zeta\}$ $(= \partial\mathcal{B}_\zeta)$ at Z, whence there is an $\tilde{\alpha} > 0$ such that $\nu^k(Z,\zeta) = \tilde{\alpha} D_k\Phi_Z(Z) = \tilde{\alpha} D_k\Phi_{(Z,\zeta)}(Z,\zeta)$

for $k = 1,2,3$. In view of (21), we must have $\alpha = \tilde{\alpha}$. Thus, α is indeed positive. As noted, the proof of (iii.4) is now complete.

To prove the equalities (4), choose $P \in \partial\mathcal{R}$ and $\zeta \in \mathbb{R}$; using the basis $\{T^j_{h1}(P)e^{(4)}_j, T^j_{h2}(P)e^{(4)}_j, e^{(4)}_4\}$ for $T_{\partial\mathcal{R}\times\mathbb{R}}(P,\zeta)$, constructed as in the proof of (iii.4) by selecting a coordinate system (U,h) in $\partial\mathcal{R}$ with $P \in U$, the definition [I.2.10.iii] says that

$$\mathcal{J}\chi^*(P,\zeta) = \frac{|D\chi^*(P,\zeta)T^j_{h1}(P)e^{(4)}_j \wedge D\chi^*(P,\zeta)T^k_{h2}(P)e^{(4)}_k \wedge D\chi^*(P,\zeta)e^{(4)}_4|}{|T^j_{h1}(P)e^{(4)}_j \wedge T^k_{h2}(P)e^{(4)}_k \wedge e^{(4)}_4|} . \tag{28}$$

For the vectors involved in the exterior product appearing in the numerator on the right of (28), we have established (14) and (15). For brevity, let us write

$$a^i_j := (\chi_\zeta \circ h^{-1})^i_{,j}(h(P)), \qquad \text{for} \quad i \in \{1,2,3\} \quad \text{and} \quad j \in \{1,2\}, \tag{29}$$

$$a^i_4 := \chi^i_{,4}(P,\zeta), \qquad \text{for} \quad i \in \{1,2,3\}. \tag{30}$$

A short computation gives

$$a^i_1 e^{(4)}_i \wedge a^j_2 e^{(4)}_j \wedge \{a^k_4 e^{(4)}_k + e^{(4)}_4\}$$

$$= \begin{vmatrix} a^1_1 & a^1_2 & a^1_4 \\ a^2_1 & a^2_2 & a^2_4 \\ a^3_1 & a^3_2 & a^3_4 \end{vmatrix} e^{(4)}_1 \wedge e^{(4)}_2 \wedge e^{(4)}_3 + \begin{vmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{vmatrix} e^{(4)}_1 \wedge e^{(4)}_2 \wedge e^{(4)}_4 \tag{31}$$

$$+ \begin{vmatrix} a^1_1 & a^1_2 \\ a^3_1 & a^3_2 \end{vmatrix} e^{(4)}_1 \wedge e^{(4)}_3 \wedge e^{(4)}_4 + \begin{vmatrix} a^2_1 & a^2_2 \\ a^3_1 & a^3_2 \end{vmatrix} e^{(4)}_2 \wedge e^{(4)}_3 \wedge e^{(4)}_4 .$$

Setting

$$A^1 := \begin{vmatrix} a_1^2 & a_1^3 \\ a_2^2 & a_2^3 \end{vmatrix}, \quad A^2 := -\begin{vmatrix} a_1^1 & a_1^3 \\ a_2^1 & a_2^3 \end{vmatrix}, \quad \text{and} \quad A^3 := \begin{vmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{vmatrix}, \tag{32}$$

we then have

$$\begin{aligned} |a_1^i e_i^{(4)} \wedge a_2^j e_j^{(4)} \wedge \{a_4^k e_k^{(4)} + e_4^{(4)}\}|^2 &= (a_4^i A^i)^2 + A^i A^i \\ &= |A|_3^2 \left\{ 1 + \left(a_4^i \frac{A^i}{|A|_3} \right)^2 \right\}. \end{aligned} \tag{33}$$

Now, clearly, using (11), for $i = 1$ or 2,

$$D\chi_\zeta(P)T_{hi}(P) = D(\chi_\zeta o h^{-1})(h(P))e_i^{(2)} = (\chi_\zeta o h^{-1})^j_{,i}(h(P))e_j^{(3)} = a_i^j e_j^{(3)}, \tag{34}$$

from which we find, accounting for (32),

$$D\chi_\zeta(P)T_{h1}(P) \times D\chi_\zeta(P)T_{h2}(P) = A^i e_i^{(3)} = A. \tag{35}$$

Since $\{D\chi_\zeta(P)T_{h1}(P),\ D\chi_\zeta(P)T_{h2}(P)\}$ is a basis for $T_{\partial\mathcal{B}_\zeta}(\chi(P,\zeta))$ (by (i.3)), it follows from (35) that $A \in N_{\partial\mathcal{B}_\zeta}(\chi(P,\zeta))$, so we have $A/|A|_3 = \nu(\chi(P,\zeta),\zeta)$ or $= -\nu(\chi(P,\zeta),\zeta)$; whichever the case,

$$(a_4^i A^i/|A|_3)^2 = \{\chi^i_{,4}(P,\zeta)\nu^i(\chi(P,\zeta),\zeta)\}^2 = \{\upsilon(\chi(P,\zeta),\zeta)\}^2. \tag{36}$$

Using (35) and (36) in (33), we find that the numerator on the right-hand side of (28) is

$$\begin{aligned} &|a_1^i e_i^{(4)} \wedge a_2^j e_j^{(4)} \wedge \{a_4^k e_k^{(4)} + e_4^{(4)}\}| \\ &= |D\chi_\zeta(P)T_{h1}(P) \times D\chi_\zeta(P)T_{h2}(P)|_3 \cdot \sqrt{\{1+\upsilon^2(\chi(P,\zeta),\zeta)\}}. \end{aligned} \tag{37}$$

Next, we compute

$$T^j_{h1}(P)e^{(4)}_j \wedge T^k_{h2}(P)e^{(4)}_k \wedge e^{(4)}_4 = \{T^1_{h1}(P)T^2_{h2}(P)-T^2_{h1}(P)T^1_{h2}(P)\}e^{(4)}_1\wedge e^{(4)}_2\wedge e^{(4)}_4$$

$$+\{T^1_{h1}(P)T^3_{h2}(P)-T^3_{h1}(P)T^1_{h2}(P)\}e^{(4)}_1\wedge e^{(4)}_3\wedge e^{(4)}_4$$

$$+\{T^2_{h1}(P)T^3_{h2}(P)-T^3_{h1}(P)T^2_{h2}(P)\}e^{(4)}_2\wedge e^{(4)}_3\wedge e^{(4)}_4,$$

whence it is easy to check that

$$|T^j_{h1}(P)e^{(4)}_j \wedge T^k_{h2}(P)e^{(4)}_k \wedge e^{(4)}_4| = |T_{h1}(P)\times T_{h2}(P)|_3. \tag{38}$$

Observing that, in this special case,

$$\hat{J}\chi(P,\zeta) := J\chi_\zeta(P) = \frac{|D\chi_\zeta(P)T_{h1}(P)\times D\chi_\zeta(P)T_{h2}(P)|_3}{|T_{h1}(P)\times T_{h2}(P)|_3},$$

insertion of (37) and (38) into (28) produces

$$J\chi^*(P,\zeta) = \sqrt{\{1+\upsilon^2(\chi(P,\zeta),\zeta)\}}\cdot\hat{J}\chi(P,\zeta),$$

just the first equality of (4). The second equality of (4) is a simple consequence of the first and the definition $\mathring{\upsilon}(P,\zeta) := \upsilon(\chi(P,\zeta),\zeta)$. Thus, (4) is correct.

Finally, the inclusion $J\chi^* \in C(\partial R\times\mathbb{R})$ shall follow from the representation (4), once it has been established that $\upsilon \in C(\partial R\times\mathbb{R})$, which we shall do shortly, in the proof of (iv) (without appealing to (iii.5)); in this regard, recall that $\hat{J}\chi \in C(\partial R\times\mathbb{R})$ (cf., [V.46.d]).

(iv) Since we now know that $\mathbb{B}^o$ is a q-regular domain, the statements here concerning $\nu_{\partial\mathbb{B}}$ $(:= \nu_{\partial\{\mathbb{B}^o\}})$ follow immediately from

Proposition [I.2.31]. Consider next $\upsilon: \partial\mathbb{B} \to \mathbb{R}$. From (3),

$$\nu^4_{\partial\mathbb{B}} = -\upsilon\cdot\{1+\upsilon^2\}^{-1/2} \quad \text{on} \quad \partial\mathbb{B},$$

from which we first see that $|\nu^4_{\partial\mathbb{B}}| < 1$ on $\partial\mathbb{B}$, then that

$$\upsilon = -\nu^4_{\partial\mathbb{B}}\cdot\{1-(\nu^4_{\partial\mathbb{B}})^2\}^{-1/2} \quad \text{on} \quad \partial\mathbb{B}. \tag{39}$$

Thus, the statements of (iv) concerning υ follow from those for $\nu_{\partial\mathbb{B}}$. Clearly, (3) allows us to use the same strategy in proving that ν possesses the properties claimed, for it shows that

$$\nu^i = \nu^i_{\partial\mathbb{B}}\cdot\{1+\upsilon^2\}^{1/2} \quad \text{on} \quad \partial\mathbb{B}. \tag{40}$$

We choose to defer the proofs of the assertions concerning $\overset{\circ}{\nu}$, $\hat{J}_\chi$, and $\overset{\circ}{\upsilon}$ until after we have verified (vi).

(v) Select $(X,t) \in \mathbb{B}^q \cup \Omega^\sigma$; we shall use the implicit function theorem to prove that $\tau(\cdot;X,t) \in C^q(\partial R)$. Let (U,h) be any coordinate system in ∂R. Since $(X,t) \notin \partial\mathbb{B}$, we see that $X \notin \partial B_t$, so $r_X(\chi(P,t)) > 0$ for each $P \in \partial R$, which gives, by [V.35.i], $\tau(P;X,t) > 0$ for each $P \in \partial R$. In particular,

$$\tau(h^{-1}(\hat{x});X,t) > 0 \quad \text{whenever} \quad \hat{x} \in h(U). \tag{41}$$

Now choose any $\hat{x}_0 \in h(U)$. Then, recalling the manner in which τ is defined,

$$r_X(\chi(h^{-1}(\hat{x}_0),\ t-\tau(h^{-1}(\hat{x}_0);X,t))) = c\tau(h^{-1}(\hat{x}_0);X,t), \tag{42}$$

whence it follows that the function $(\hat{x},\zeta) \mapsto r_X(\chi(h^{-1}(\hat{x}),t-\zeta))$ on

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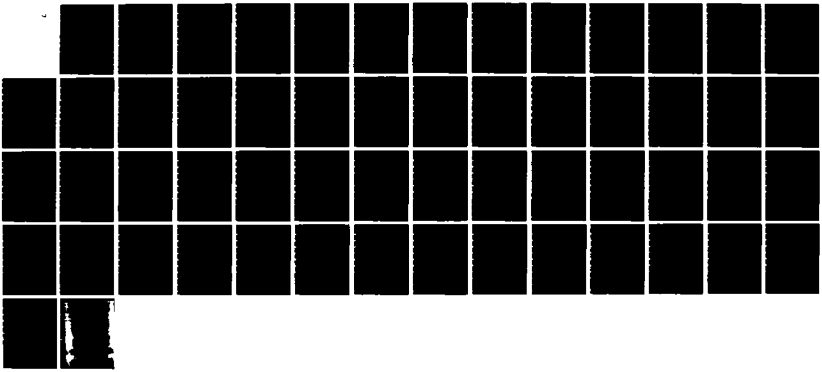

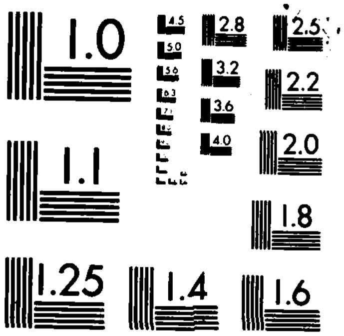

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$h(U)\times\mathbb{R}$ is positive at $(\hat{x}_0,\tau(h^{-1}(\hat{x}_0);X,t))$; the function obviously being continuous on $h(U)\times\mathbb{R}$, we therefore conclude that there exist an open neighborhood $V_0 \subset h(U)$ of $\hat{x}_0$ and a $\delta_0 > 0$ such that

$$r_X(\chi(h^{-1}(\hat{x}),t-\zeta)) > 0 \quad \text{for} \quad \hat{x} \in V_0, \quad |\zeta-\tau(h^{-1}(\hat{x}_0);X,t)| < \delta_0. \tag{43}$$

Since $\chi \in C^q(\partial R\times\mathbb{R};\mathbb{R}^3)$ implies that $(\hat{x},s) \mapsto \chi(h^{-1}(\hat{x}),s) = \chi o h^{*-1}(\hat{x},s)$ is in $C^q(h(U)\times\mathbb{R};\mathbb{R}^3)$, the same is clearly true of the function $(\hat{x},\zeta) \mapsto \chi(h^{-1}(\hat{x}),t-\zeta)$. The latter fact, when coupled with (43), tells us that the map $(\hat{x},\zeta) \mapsto r_X(\chi(h^{-1}(\hat{x}),t-\zeta))$ is in $C^q(V_0\times(\alpha,\beta))$, where $\alpha := \tau(h^{-1}(\hat{x}_0);X,t)-\delta_0$, $\beta := \tau(h^{-1}(\hat{x}_0);X,t)+\delta_0$. Defining F: $V_0\times(\alpha,\beta) \to \mathbb{R}$ by

$$F(\hat{x},\zeta) := \zeta- \frac{1}{c}\, r_X(\chi(h^{-1}(\hat{x}),t-\zeta)), \quad \text{for} \quad \hat{x} \in V_0, \quad \zeta \in (\alpha,\beta), \tag{44}$$

we have $F \in C^q(V_0\times(\zeta,\beta))$ and $F(\hat{x}_0,\tau(h^{-1}(\hat{x}_0);X,t)) = 0$ (by (42)). Moreover, the Cauchy-Schwarz inequality gives

$$\begin{aligned} &F_{,3}(\hat{x}_0,\tau(h^{-1}(\hat{x}_0);X,t)) \\ &= 1+ \frac{1}{c} \operatorname{grad} r_X(\chi(h^{-1}(\hat{x}_0),t-\tau(h^{-1}(\hat{x}_0);X,t)))\bullet\chi_{,4}(h^{-1}(\hat{x}_0),t-\tau(h^{-1}(\hat{x}_0);X,t)) \\ &\geq 1- \frac{1}{c}\, |\chi_{,4}(h^{-1}(\hat{x}_0),t-\tau(h^{-1}(\hat{x}_0);X,t))|_3 \geq 1- \frac{c^*}{c} > 0, \end{aligned} \tag{45}$$

having used (V.46.3). The implicit function theorem and its proof (cf., *e.g.*, [VI.2]) then imply that there exist an open neighborhood $W_0 \subset V_0$ of $\hat{x}_0$ and a unique function $f \in C^q(W_0)$ such that $(\hat{x},f(\hat{x})) \in V_0\times(\alpha,\beta)$ for each $\hat{x} \in W_0$, $f(\hat{x}_0) = \tau(h^{-1}(\hat{x}_0);X,t)$, and $f(\hat{x})- \frac{1}{c}\, r_X(\chi(h^{-1}(\hat{x}),t-f(\hat{x}))) = F(\hat{x},f(\hat{x})) = 0$ for each $\hat{x} \in W_0$. The uniqueness of τ (cf., [V.31]) implies, of course, that we must have

$$f(\hat{x}) = \tau(h^{-1}(\hat{x});X,t) \quad \text{for each} \quad \hat{x} \in W_0,$$

so the function $\hat{x} \mapsto \tau(h^{-1}(\hat{x});X,t)$ is in $C^q(W_0)$. Since $\hat{x}_0$ was an arbitrary point of $h(U)$, we can now assert that $\tau(h^{-1}(\cdot);X,t) \in C^q(h(U))$, hence that $\tau(\cdot;X,t) \in C^q(\partial R)$.

The proof of (5) is now straightforward: again with $(X,t) \in \mathbf{B}^q \cup \Omega^\sigma$ and (U,h) any coordinate system in ∂R, let us temporarily write $\tau_h(\hat{x}) := \tau(h^{-1}(\hat{x});X,t)$, for each $\hat{x} \in h(U)$. Then

$$\tau_h(\hat{x}) - \frac{1}{c} r_X(\chi(h^{-1}(\hat{x}), t-\tau_h(\hat{x}))) = 0 \quad \text{for each} \quad \hat{x} \in h(U), \tag{46}$$

and $\tau_h \in C^q(h(U))$. Consider the function $[\chi]_{(X,t)} \circ h^{-1}$: $h(U) \to \mathbb{R}^3$, given more explicitly by

$$\begin{aligned} [\chi]_{(X,t)} \circ h^{-1}(\hat{x}) &= \chi(h^{-1}(\hat{x}), t-\tau(h^{-1}(\hat{x});X,t)) \\ &= \chi(h^{-1}(\hat{x}), t-\tau_h(\hat{x})), \quad \text{for each} \quad \hat{x} \in h(U). \end{aligned} \tag{47}$$

This is the composition of the map $(\hat{x},\zeta) \mapsto \chi \circ h^{*-1}(\hat{x},\zeta) = \chi(h^{-1}(\hat{x}),\zeta)$, in $C^q(h(U)\times\mathbb{R};\mathbb{R}^3)$, with $\hat{x} \mapsto (\hat{x}, t-\tau(h^{-1}(\hat{x});X,t))$, in $C^q(h(U);\mathbb{R}^3)$, showing that $[\chi]_{(X,t)} \circ h^{-1} \in C^q(h(U);\mathbb{R}^3)$. A short computation, using the composite function theorem, yields

$$([\chi]_{(X,t)} \circ h^{-1})^j_{,i}(\hat{x}) = (\chi \circ h^{*-1})^j_{,i}(\hat{x}, t-\tau_h(\hat{x}))$$
$$-\tau_{h,i}(\hat{x}) \cdot (\chi \circ h^{*-1})^j_{,3}(\hat{x}, t-\tau_h(\hat{x}))$$
$$= (\chi_{t-\tau_h(\hat{x})} \circ h^{-1})^j_{,i}(\hat{x}) - \tau_{h,i}(\hat{x}) \cdot \chi^j_{,4}(h^{-1}(\hat{x}), t-\tau_h(\hat{x})), \tag{48}$$

for $\hat{x} \in h(U)$, $j \in \{1,2,3\}$, and $i \in \{1,2\}$,

the second equality following from (V.36.2) and the fact that $(\chi \circ h^{*-1})(\hat{x}, \zeta) = (\chi_\zeta \circ h^{-1})(\hat{x})$ for $\zeta \in \mathbb{R}$, $\hat{x} \in h(U)$. We can now differentiate in (46), obtaining, *via* (48) and some rearrangement,

$$\{1 + r_{X,j}(\chi(h^{-1}(\hat{x}), t-\tau_h(\hat{x}))) \cdot \chi^{j}_{,4}{}^{c}(h^{-1}(\hat{x}), t-\tau_h(\hat{x}))\}\tau_{h,i}(\hat{x})$$
$$= \frac{1}{c}\, r_{X,k}(\chi(h^{-1}(\hat{x}), t-\tau_h(\hat{x}))) \cdot (\chi_{t-\tau_h(\hat{x})} \circ h^{-1})^k_{,i}(\hat{x}), \tag{49}$$

for $\hat{x} \in h(U)$ and $i \in \{1,2\}$.

Reasoning just as in the derivation of (45), we find that the coefficient of $\tau_{h,i}(\hat{x})$ on the left of (49) is $\geq 1 - \frac{c^*}{c} > 0$. Thus, (49) immediately gives (5), upon introducing the retardation notation. The proof of (v) is complete.

(vi) Fix $(X,t) \in \mathbb{B}^o \cup \Omega^\sigma$: Proposition [V.38] says that $[\chi]_{(X,t)}: \partial R \to \mathbb{R}^3$ is a homeomorphism of ∂R onto $\partial\mathbb{B}(X,t)$ (for any $(X,t) \in \mathbb{R}^4$), while it was shown, in the course of the proof of (v), that $[\chi]_{(X,t)} \circ h^{-1} \in C^q(h(U);\mathbb{R}^3)$ whenever (U,h) is a coordinate system in ∂R, *i.e.*, that $[\chi]_{(X,t)} \in C^q(\partial R;\mathbb{R}^3)$ (here, the condition $(X,t) \in \mathbb{B}^q \cup \Omega^\sigma$ is, in general, necessary). To complete the proof of the contention that $[\chi]_{(X,t)}$ is a q-imbedding, we must show that the rank of $[\chi]_{(X,t)}$ at each $P \in \partial R$ is 2, or, equivalently, that $J[\chi]_{(X,t)}(P) > 0$ for each $P \in \partial R$, which shall be apparent once the explicit representation (7) has been verified. Indeed, we have $\hat{J}\chi > 0$ on $\partial R \times \mathbb{R}$ (cf., (V.46.7)), so $[\hat{J}\chi]_{(X,t)}(P) > 0$ for each $P \in \partial R$, while the result (V.42.2) shows that $1-\tau_{;4}(P;X,t) > 0$, since $1+ \text{grad}\, r_X([\chi]_{(X,t)}(P)) \bullet [\chi^c_{;4}]_{(X,t)}(P) \geq 1- \frac{c^*}{c} > 0$ for each $P \in \partial R$ (just as in the proof of (v), the inclusion $(X,t) \in \mathbb{B}^{\circ} \cup \Omega^{\sigma}$ gives $\tau(\cdot;X,t) > 0$ on ∂R, so $r_X([\chi]_{(X,t)}(\cdot)) > 0$ on ∂R, and $\text{grad}\, r_X([\chi]_{(X,t)}(\cdot))$ is defined on ∂R). Further, the definition (2) and inequality (V.46.3) give $|\upsilon| \leq c^*$ on $\partial \mathbb{B}$, whence $|\overset{\circ}{\upsilon}{}^c| \leq \frac{c^*}{c} < 1$ on $\partial R \times \mathbb{R}$, and we have

$$
\begin{aligned}
&|[\overset{\circ}{\upsilon}]_{(X,t)}(P)+[\overset{\circ}{\upsilon}{}^c]_{(X,t)}(P) \cdot \text{grad}\, r_X([\chi]_{(X,t)}(P))|_3 \\
&= \{1+2[\overset{\circ}{\upsilon}{}^c]_{(X,t)}(P)\cdot[\overset{\circ}{\upsilon}]_{(X,t)}(P) \bullet \text{grad}\, r_X([\chi]_{(X,t)}(P))+[\overset{\circ}{\upsilon}{}^c]^2_{(X,t)}(P)\}^{1/2} \\
&\geq \{1-2|[\overset{\circ}{\upsilon}{}^c]_{(X,t)}(P)|+[\overset{\circ}{\upsilon}{}^c]^2_{(X,t)}(P)\}^{1/2} = 1-|[\overset{\circ}{\upsilon}{}^c]_{(X,t)}(P)| > 0,
\end{aligned}
\tag{50}
$$

for each $P \in \partial R$. Therefore, proving (7) shall also finish the proof of (vi.1).

Turning, then, to (vi.5), choose $P \in \partial R$ and a coordinate system (U,h) in ∂R with $P \in U$. Once again selecting the basis $\{T_{hi}(P)\}_{i=1}^2$ for $T_{\partial R}(P)$, given by (11), we can compute $J[\chi]_{(X,t)}(P)$ in the present case from

$$J[\chi]_{(X,t)}(P) = \frac{|D[\chi]_{(X,t)}(P)T_{h1}(P)\times D[\chi]_{(X,t)}(P)T_{h2}(P)|_3}{|T_{h1}(P)\times T_{h2}(P)|_3} . \tag{51}$$

Now,

$$D[\chi]_{(X,t)}(P)T_{hi}(P) = D([\chi]_{(X,t)}\circ h^{-1})(h(P))e_i^{(2)}$$
$$= ([\chi]_{(X,t)}\circ h^{-1})^j_{,i}(h(P))e_j^{(3)}, \quad \text{for} \quad i \in \{1,2\}, \tag{52}$$

while we have already available the expressions (48) (in which we set $\hat{x} = h(P)$) for $([\chi]_{(X,t)}\circ h^{-1})^j_{,i}(h(P))$, for $j \in \{1,2,3\}$, $i \in \{1,2\}$. Thus, for $i \in \{1,2,3\}$,

$$\{D[\chi]_{(X,t)}(P)T_{h1}(P)\times D[\chi]_{(X,t)}(P)T_{h2}(P)\}^i$$
$$= \varepsilon_{ijk}\{(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,1}(h(P))-\tau(h^{-1}(\cdot);X,t)_{,1}(h(P))\cdot[\chi^j_{;4}]_{(X,t)}(P)\}$$
$$\cdot\{(\chi_{t-\tau(P;X,t)}\circ h^{-1})^k_{,2}(h(P))-\tau(h^{-1}(\cdot);X,t)_{,2}(h(P))\cdot[\chi^k_{;4}]_{(X,t)}(P)\}$$
$$= \varepsilon_{ijk}(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,1}(h(P))\cdot(\chi_{t-\tau(P\cdot X,t)}\circ h^{-1})^k_{,2}(h(P)) \tag{53}$$
$$-\tau(h^{-1}(\cdot);X,t)_{,2}(h(P))\cdot\varepsilon_{ijk}(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,1}(h(P))\cdot[\chi^k_{;4}]_{(X,t)}(P)$$
$$+\tau(h^{-1}(\cdot);X,t)_{,1}(h(P))\cdot\varepsilon_{ijk}(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,2}(h(P))\cdot[\chi^k_{;4}]_{(X,t)}(P);$$

replacing $\tau(h^{-1}(\cdot);X,t)_{,\ell}(h(P))$, for $\ell = 1,2$, by the expressions given in (5) (with $\hat{x} = h(P)$), and taking into account (V.42.2), the right-hand side of (53) is

$$= \varepsilon_{ijk}(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,1}(h(P))\cdot(\chi_{t-\tau(P;X,t)}\circ h^{-1})^k_{,2}(h(P))$$
$$+\{1-\tau_{;4}(P;X,t)\}\cdot\varepsilon_{ijk}\{(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,2}(h(P))$$
$$\cdot(\chi_{t-\tau(P;X,t)}\circ h^{-1})^u_{,1}(h(P))-(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,1}(h(P))$$
$$\cdot(\chi_{t-\tau(P;X,t)}\circ h^{-1})^u_{,2}(h(P))\}\cdot r_{X,u}([\chi]_{(X,t)}(P))\cdot[\chi^{k^c}_{;4}]_{(X,t)}(P)$$

$$= \varepsilon_{ijk}(\chi_{t-\tau(P;X,t)} \circ h^{-1})^{j}_{,1}(h(P)) \cdot (\chi_{t-\tau(P;X,t)} \circ h^{-1})^{k}_{,2}(h(P))$$

$$-\{1-\tau_{;4}(P;X,t)\} \cdot \varepsilon_{\ell mn}(\chi_{t-\tau(P;X,t)} \circ h^{-1})^{m}_{,1}(h(P)) \tag{54}$$

$$\cdot (\chi_{t-\tau(P;X,t)} \circ h^{-1})^{n}_{,2}(h(P)) \cdot \{\delta_{i\ell} r_{X,k}([\chi]_{(X,t)}(P)) \cdot [\chi^{k^c}_{,4}]_{(X,t)}(P)$$

$$-[\chi^{\ell^c}_{,4}]_{(X,t)}(P) \cdot r_{X,i}([\chi]_{(X,t)}(P))\},$$

the latter equality following from

$$(\chi_{t-\tau(P;X,t)} \circ h^{-1})^{j}_{,2}(h(P)) \cdot (\chi_{t-\tau(P;X,t)} \circ h^{-1})^{u}_{,1}(h(P))$$

$$-(\chi_{t-\tau(P;X,t)} \circ h^{-1})^{j}_{,1}(h(P)) \cdot (\chi_{t-\tau(P;X,t)} \circ h^{-1})^{u}_{,2}(h(P))$$

$$= -\varepsilon_{ju\ell}\varepsilon_{\ell mn}(\chi_{t-\tau(P;X,t)} \circ h^{-1})^{m}_{,1}(h(P)) \cdot (\chi_{t-\tau(P;X,t)} \circ h^{-1})^{n}_{,2}(h(P)),$$

and then

$$\varepsilon_{ijk}\varepsilon_{ju\ell} r_{X,u}([\chi]_{(X,t)}(P)) \cdot [\chi^{k^c}_{,4}]_{(X,t)}(P)$$

$$= \delta_{i\ell} r_{X,k}([\chi]_{(X,t)}(P)) \cdot [\chi^{k^c}_{,4}]_{(X,t)}(P) - r_{X,i}([\chi]_{(X,t)}(P)) \cdot [\chi^{\ell^c}_{,4}]_{(X,t)}(P).$$

Once again, it is to be emphasized that grad $r_X([\chi]_{(X,t)}(P))$ exists, since $r_X([\chi]_{(X,t)}(P)) = c\tau(P;X,t) > 0$, due to the hypothesis $(X,t) \in \mathbf{B}^{\sigma} \cup \Omega^{\sigma}$. To develop further the expression given in (54), we use (V.42.3) to rewrite it as

$$= \{1-\{1-\tau;_4(P;X,t)\}\cdot r_{X,\ell}([\chi]_{(X,t)}(P))\cdot[\chi^{\ell^c}_{,4}]_{(X,t)}(P)\}$$

$$\cdot\varepsilon_{ijk}(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,1}(h(P))\cdot(\chi_{t-\tau(P;X,t)}\circ h^{-1})^k_{,2}(h(P))$$

$$+\{1-\tau;_4(P;X,t)\}\cdot[\chi^{\ell^c}_{,4}]_{(X,t)}(P)\cdot r_{X,i}([\chi]_{(X,t)}(P))$$

$$\cdot\varepsilon_{\ell jk}(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,1}(h(P))\cdot(\chi_{t-\tau(P;X,t)}\circ h^{-1})^k_{,2}(h(P)) \tag{55}$$

$$= \{1-\tau;_4(P;X,t)\}\,\cdot$$

$$\cdot\{\varepsilon_{ijk}(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,1}(h(P))\cdot(\chi_{t-\tau(P;X,t)}\circ h^{-1})^k_{,2}(h(P))$$

$$+\{[\chi^{\ell^c}_{,4}]_{(X,t)}(P)\cdot\varepsilon_{\ell jk}(\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,1}(h(P))$$

$$\cdot(\chi_{t-\tau(P;X,t)}\circ h^{-1})^k_{,2}(h(P))\}r_{X,i}([\chi]_{(X,t)}(P))\}.$$

Now, note that

$$D\chi_{t-\tau(P;X,t)}(P)T_{hi}(P) = D(\chi_{t-\tau(P;X,t)}\circ h^{-1})(h(P))e^{(2)}_i$$
$$= (\chi_{t-\tau(P;X,t)}\circ h^{-1})^j_{,i}(h(P))e^{(3)}_j, \quad i\in\{1,2\}. \tag{56}$$

Further, $\{D\chi_{t-\tau(P;X,t)}(P)T_{hi}(P)\}^2_{i=1}$ being, by (i.3), a basis for $T_{\partial\mathcal{B}_{t-\tau(P;X,t)}}(\chi(P,t-\tau(P;X,t)))$, we must have

$$[\overset{\circ}{\nu}]_{(X,t)}(P) := \overset{\circ}{\nu}(P,t-\tau(P;X,t)) := \nu(\chi(P,t-\tau(P;X,t)),t-\tau(P;X,t))$$

$$:= \nu_{\partial\mathcal{B}_{t-\tau(P;X,t)}}(\chi(P,t-\tau(P;X,t))) \tag{57}$$

$$= \iota\,\frac{D\chi_{t-\tau(P;X,t)}(P)T_{h1}(P)\times D\chi_{t-\tau(P;X,t)}(P)T_{h2}(P)}{|D\chi_{t-\tau(P;X,t)}(P)T_{h1}(P)\times D\chi_{t-\tau(P;X,t)}(P)T_{h2}(P)|_3},$$

where ι is $+1$ or -1, as the case may be. Since

$$[\overset{\circ}{\upsilon}]_{(X,t)}(P) := \overset{\circ}{\upsilon}(P,t-\tau(P;X,t)) := \upsilon(\chi(P,t-\tau(P;X,t)),t-\tau(P;X,t))$$

$$:= \nu(\chi(P,t-\tau(P;X,t)),t-\tau(P;X,t))$$

$$\bullet\chi,_4(\chi^{-1}_{t-\tau(P;X,t)}(\chi(P,t-\tau(P;X,t))),t-\tau(P;X,t)) \tag{58}$$

$$= [\overset{\circ}{\nu}]_{(X,t)}(P)\bullet\chi,_4(P,t-\tau(P;X,t))$$

$$= [\overset{\circ}{\nu}]_{(X,t)}(P)\bullet[\chi,_4]_{(X,t)}(P),$$

we obtain from (55), using also (56) and (57),

$$D[\chi]_{(X,t)}(P)T_{h1}(P)\times D[\chi]_{(X,t)}(P)T_{h2}(P)$$

$$= \iota|D\chi_{t-\tau(P;X,t)}(P)T_{h1}(P)\times D\chi_{t-\tau(P;X,t)}(P)T_{h2}(P)|_3\cdot\{1-\tau;_4(P;X,t)\} \tag{59}$$

$$\cdot\{[\overset{\circ}{\nu}]_{(X,t)}(P)+[\overset{\circ}{\upsilon}^c]_{(X,t)}(P)\cdot\text{grad }\tau_X([\chi]_{(X,t)}(P))\}.$$

Insert the expression appearing on the right of (59) into (51); (7) results thereby, for, $|\iota| = 1$, we have already shown that $\{1-\tau;_4(P;X,t)\} > 0$, and

$$[\hat{J}\chi]_{(X,t)}(P) := \hat{J}\chi(P,t-\tau(P;X,t)) := J\chi_{t-\tau(P;X,t)}(P)$$

$$= \frac{|D\chi_{t-\tau(P;X,t)}(P)T_{h1}(P)\times D\chi_{t-\tau(P;X,t)}(P)T_{h2}(P)|_3}{|T_{h1}(P)\times T_{h2}(P)|_3}. \tag{60}$$

With this, the proofs of (vi.1) and (vi.5) have been completed.

Statement (vi.2) is proven by the same reasoning used to verify (i.2) and (iii.2): by [I.2.17.i] and (vi.1), $\partial\mathbb{B}(X,t) = [\chi]_{(X,t)}(\partial R)$ is a (2,3;q)-manifold. According to Theorem [V.32], $\mathbb{B}(X,t)$ is regularly closed; in particular, $\partial\{\mathbb{B}(X,t)^{o}\} = \partial\mathbb{B}(X,t)$, $\mathbb{B}(X,t)^{o-\prime} = \mathbb{B}(X,t)' = \mathbb{B}'(X,t) = \Omega^{\sigma}(X,t)$, and $\mathbb{B}(X,t)^{o}$ is regularly open. Directly from Proposition [I.2.29], $\mathbb{B}(X,t)^{o}$ and $\Omega^{\sigma}(X,t)$ are q-regular domains in $\mathbb{R}^3$.

(vi.3) is a simple consequence of (vi.1) and [I.2.17.ii].

Turning finally to (vi.4), let $P \in \partial R$. Parenthetically, we remark that it is easy to see from (59) that $[\overset{o}{\upsilon}]_{(X,t)}(P)+[\overset{oc}{\upsilon}]_{(X,t)}(P)\cdot \operatorname{grad} r_X([X]_{(X,t)}(P)) \in N_{\partial\{\mathbb{B}(X,t)^{o}\}}([X]_{(X,t)}(P))$, and it was shown in the proof of (vi.5) that this vector is non-zero; to prove that it provides an *exterior* normal to $\partial\{\mathbb{B}(X,t)^{o}\}$ at $[X]_{(X,t)}(P)$, however, it appears to be easiest to proceed as follows: because $(X,t) \notin \partial\mathbb{B} = \partial\{\mathbb{B}^{o}\}$, $([X]_{(X,t)}(P),t-\tau(P;X,t)) \in \partial\mathcal{B}_{t-\tau(P;X,t)}\times\{t-\tau(P;X,t)\} \subset \partial\mathbb{B} = \partial\{\mathbb{B}^{o}\}$, and $\mathbb{B}^{o}$ is a q-regular domain, it follows that there exist an $\varepsilon > 0$ and, setting $W_\varepsilon := B^4_\varepsilon([X]_{(X,t)}(P),t-\tau(P;X,t))$, a function $\Phi \in C^q(W_\varepsilon)$ such that $(X,t) \notin W_\varepsilon$, $\operatorname{grad}\Phi(Y,s) \neq 0$ for each $(Y,s) \in W_\varepsilon$,

$$\partial\{\mathbb{B}^{o}\}\cap W_\varepsilon = \{(Y,s) \in W_\varepsilon \mid \Phi(Y,s) = 0\}, \tag{61}$$

and

$$\mathbb{B}^{o}\cap W_\varepsilon = \{(Y,s) \in W_\varepsilon \mid \Phi(Y,s) < 0\}. \tag{62}$$

Then $\operatorname{grad}\Phi(Z,\zeta)$ is an exterior normal to $\partial\{\mathbb{B}^{o}\}$ at each $(Z,\zeta) \in \partial\{\mathbb{B}^{o}\}\cap W_\varepsilon$; in particular, using (3), it is easy to see that $\Phi_{,\ell}([X]_{(X,t)}(P),t-\tau(P;X,t)) \neq 0$ for some $\ell \in \{1,2,3\}$. Consequently, by returning and choosing a smaller positive number, if necessary, we can suppose that ε is such that

$$\Phi_{,\ell}(Y,s) \neq 0, \quad \text{for each} \quad (Y,s) \in W_\varepsilon. \tag{63}$$

Next, following the construction in the proof of (iii.4), define $\hat{\Phi}\colon B^3_\varepsilon([\chi]_{(X,t)}(P)) \to \mathbb{R}$ *via*

$$\hat{\Phi}(Y) := \Phi(Y,t-\tau(P;X,t)), \quad \text{for each} \quad Y \in B^3_\varepsilon([\chi]_{(X,t)}(P)). \tag{64}$$

Obviously, $\hat{\Phi} \in C^q(B^3_\varepsilon([\chi]_{(X,t)}(P)))$, with

$$\begin{gathered} \operatorname{grad} \hat{\Phi}(Y) = \Phi_{,i}(Y,t-\tau(P;X,t))e^{(3)}_i \neq 0, \\ \text{for each} \quad Y \in B^3_\varepsilon([\chi]_{(X,t)}(P)), \end{gathered} \tag{65}$$

having taken into account (63). Reasoning as in the proof of (iii.4), it is easy to show, using (61), (62), (64), [V.15.ii], and [V.15.iii], that $\partial\{B^o_{t-\tau(P;X,t)}\}\cap B^3_\varepsilon([\chi]_{(X,t)}(P)) = \{Y \in B^3_\varepsilon([\chi]_{(X,t)}(P)) \mid \hat{\Phi}(Y) = 0\}$, and $B^o_{t-\tau(P;X,t)}\cap B^3_\varepsilon([\chi]_{(X,t)}(P)) = \{Y \in B^3_\varepsilon([\chi]_{(X,t)}(P)) \mid \hat{\Phi}(Y) < 0\}$, remembering that $\partial\{\mathbb{B}^o\} = \partial\mathbb{B}$ and $\partial\{B^o_\zeta\} = \partial B_\zeta$ for each $\zeta \in \mathbb{R}$. Thus, $\operatorname{grad} \hat{\Phi}([\chi]_{(X,t)}(P)) = \Phi_{,i}([\chi]_{(X,t)}(P),t-\tau(P;X,t))e^{(3)}_i$ is an exterior normal for $\partial\{B^o_{t-\tau(P;X,t)}\}$ at $[\chi]_{(X,t)}(P)$, whence there exists an $\hat{\alpha} > 0$ such that

$$\Phi_{,i}([\chi]_{(X,t)}(P),t-\tau(P;X,t))e^{(3)}_i = \hat{\alpha}\cdot[\overset{\circ}{\nu}]_{(X,t)}(P). \tag{66}$$

Now, the map $s \mapsto (\chi(P,t-\tau(P;X,t)+s),t-\tau(P;X,t)+s)$, on $\mathbb{R}$ into $\mathbb{R}^4$, is continuous and has values on $\partial\mathbb{B}$ (cf., [V.15.iii]). Since W_ε is an open neighborhood of $(\chi(P,t-\tau(P;X,t)),t-\tau(P;X,t))$, there is an

$\varepsilon_0 > 0$ for which $(\chi(P,t-\tau(P;X,t)+s),t-\tau(P;X,t)+s) \in \partial\{B^o\}\cap W_\varepsilon$ if $|s| < \varepsilon_0$, so

$$\Phi(\chi(P,t-\tau(P;X,t)+s),t-\tau(P;X,t)+s) = 0, \quad \text{for} \quad |s| < \varepsilon_0, \tag{67}$$

by (61). Differentiating and setting $s = 0$, (67) yields

$$\Phi,_i([\chi]_{(X,t)}(P),t-\tau(P;X,t))\cdot[\chi^i,_4]_{(X,t)}(P)+\Phi,_4([\chi]_{(X,t)}(P),t-\tau(P;X,t))$$
$$= 0,$$

which, with (66), gives, in turn,

$$\begin{aligned}\Phi,_4([\chi]_{(X,t)}(P),t-\tau(P;X,t)) &= -\hat{\alpha}[\overset{o}{\nu}]_{(X,t)}(P)\bullet[\chi,_4]_{(X,t)}(P) \\ &= -\hat{\alpha}[\overset{o}{\nu}]_{(X,t)}(P),\end{aligned} \tag{68}$$

the latter equality by (58). Having laid the groundwork for the proof of (vi.4), let us continue by observing that $W_\varepsilon(X,t)$ is open in $\mathbb{R}^3$ (by [V.20.i]) and contains $[\chi]_{(X,t)}(P)$, since

$$([\chi]_{(X,t)}(P),\ t-\frac{1}{c}\,r_X([\chi]_{(X,t)}(P))) = ([\chi]_{(X,t)}(P),t-\tau(P;X,t)) \in W_\varepsilon.$$

Consider the function $[\Phi]_{[X,t]}\colon\ W_\varepsilon(X,t) \to \mathbb{R}$ (cf., [V.36.i]), given explicitly by

$$[\Phi]_{[X,t]}(Y) := \Phi(Y,\ t-\frac{1}{c}\,r_X(Y)), \quad \text{for} \quad Y \in W_\varepsilon(X,t).$$

Since $X \notin W_\varepsilon(X,t)$ (because we ensured that $(X,t) \notin W_\varepsilon$), it follows that $[\Phi]_{[X,t]} \in C^q(W_\varepsilon(X,t))$, with, for $Y \in W_\varepsilon(X,t)$ and $i \in \{1,2,3\}$,

$$[\Phi]_{[X,t],i}(Y) = \Phi,_i(Y,\ t-\frac{1}{c}\,r_X(Y))-\Phi,_4(Y,\ t-\frac{1}{c}\,r_X(Y))\cdot\frac{1}{c}\,r_{X,i}(Y).$$

Thus, using (66) and (68),

$$\begin{aligned}\operatorname{grad}\,[\Phi]_{[X,t]}([\chi]_{(X,t)}(P)) &= \Phi_{,i}([\chi]_{(X,t)}(P),t-\tau(P;X,t))e_i^{(3)} \\ &\quad -\Phi_{,4}([\chi]_{(X,t)}(P),t-\tau(P;X,t)) \\ &\quad \cdot \frac{1}{c}\operatorname{grad} r_X([\chi]_{(X,t)}(P)) \qquad (69)\\ &= \hat{\alpha}\{[\overset{o}{\upsilon}]_{(X,t)}(P)+[\overset{o}{\upsilon}{}^{c}]_{(X,t)}(P) \\ &\quad \cdot \operatorname{grad} r_X([\chi]_{(X,t)}(P))\}.\end{aligned}$$

In particular, (69) implies that $\operatorname{grad}\,[\Phi]_{[X,t]}([\chi]_{(X,t)}(P)) \neq 0$, so we can find an $\tilde{\varepsilon} > 0$ such that $B^3_{\tilde{\varepsilon}}([\chi]_{(X,t)}(P)) \subset W_\varepsilon(X,t)$, and $\operatorname{grad}\,[\Phi]_{[X,t]}(Y) \neq 0$ for each $Y \in B^3_{\tilde{\varepsilon}}([\chi]_{(X,t)}(P))$. Let us next point out that

$$\partial\{\mathbb{B}(X,t)^o\}\cap W_\varepsilon(X,t) = \{Y \in W_\varepsilon(X,t)\,|\;\; [\Phi]_{[X,t]}(Y) = 0\}, \qquad (70)$$

and

$$\mathbb{B}(X,t)^o\cap W_\varepsilon(X,t) = \{Y \in W_\varepsilon(X,t)\,|\;\; [\Phi]_{[X,t]}(Y) < 0\}. \qquad (71)$$

To prove (70), suppose first that $Y \in \partial\{\mathbb{B}(X,t)^o\}\cap W_\varepsilon(X,t) = \partial\mathbb{B}(X,t)\cap W_\varepsilon(X,t)$: by [V.32.iii], we have $Y \in \partial\mathcal{B}_{t-\frac{1}{c}r_X(Y)}$, and so, using [V.15.iii] and the definition of $W_\varepsilon(X,t)$, $(Y,t-\frac{1}{c}r_X(Y)) \in \partial\mathbf{B}\cap W_\varepsilon = \partial\{\mathbf{B}^o\}\cap W_\varepsilon$. Thus, in view of (61), $[\Phi]_{[X,t]}(Y) = \Phi(Y,t-\frac{1}{c}r_X(Y)) = 0$. To prove the reversed inclusion, let $Y \in W_\varepsilon(X,t)$, with $[\Phi]_{[X,t]}(Y) = 0$: then $(Y,t-\frac{1}{c}r_X(Y)) \in W_\varepsilon$ and $\Phi(Y,t-\frac{1}{c}r_X(Y)) = 0$, whence (61) shows that $(Y,t-\frac{1}{c}r_X(Y)) \in \partial\{\mathbf{B}^o\}\cap W_\varepsilon$, so

$Y \in \{\partial\{\mathbb{B}^{o}\}\cap W_{\varepsilon}\}(X,t) = \partial\mathbb{B}(X,t)\cap W_{\varepsilon}(X,t) = \partial\{\mathbb{B}(X,t)^{o}\}\cap W_{\varepsilon}(X,t)$. Therefore, (70) is true; (71) is proven in the same manner, using [V.32.iv], [V.15.ii], and (62) at the appropriate junctures. Now, clearly (70) and (71) continue to hold if $W_{\varepsilon}(X,t)$ is replaced in them by $B^3_{\varepsilon}([\chi]_{(X,t)}(P))$, and we can conclude that grad $[\phi]_{[X,t]}([\chi]_{(X,t)}(P))$ is an exterior normal for $\partial\{\mathbb{B}(X,t)^{o}\}$ at $[\chi]_{(X,t)}(P)$; recalling (69) and the fact that $\hat{\alpha} > 0$, the same must be true of $[\overset{o}{\nu}]_{(X,t)}(P)+[\overset{oc}{\upsilon}]_{(X,t)}(P)\cdot$ grad $r_X([\chi]_{(X,t)}(P))$. Statement (vi.4) has been proven.

(iv) (conclusion) Let $q \geq 2$: we must still show that $D^k_4\overset{o}{\nu} \in C(\partial\mathcal{R}\times\mathbb{R};\mathbb{R}^3)$, and $D^k_4\hat{J}\chi$ and $D^k_4\overset{o}{\upsilon}$ are in $C(\partial\mathcal{R}\times\mathbb{R})$, $k = 1,\ldots,q-1$ or for all k if $q = \infty$.[†] Choose $P \in \partial\mathcal{R}$, $\zeta \in \mathbb{R}$, and a coordinate system (U,h) in $\partial\mathcal{R}$ with $P \in U$. Define $\overset{o}{N}$: $U\times\mathbb{R} \to \mathbb{R}^3$ by setting

$$\overset{o}{N}{}^{i}(\tilde{P},\tilde{\zeta}) := \varepsilon_{ijk}\{(\chi_{\tilde{\zeta}}\circ h^{-1})^{j}_{,1}(h(\tilde{P}))\}\{(\chi_{\tilde{\zeta}}\circ h^{-1})^{k}_{,2}(h(\tilde{P}))\}$$

$$= \varepsilon_{ijk}\{(\chi\circ h^{*-1})^{j}_{,1}(h(\tilde{P}),\tilde{\zeta})\}\{(\chi\circ h^{*-1})^{k}_{,2}(h(\tilde{P}),\tilde{\zeta})\} \qquad (72)$$

for $\tilde{P} \in U$ and $\tilde{\zeta} \in \mathbb{R}$.

Because $\chi\circ h^{*-1} \in C^q(h(U)\times\mathbb{R};\mathbb{R}^3)$, it is obvious that $D^k_4\overset{o}{N} \in C(U\times\mathbb{R};\mathbb{R}^3)$ for $k = 1,\ldots,q-1$, or all k if $q = \infty$. By reasoning as in the proofs of (iii) and (vi), we can deduce that

$$\overset{o}{\nu} = \iota\,\frac{\overset{o}{N}}{|\overset{o}{N}|_3} \quad \text{in} \quad U\times\mathbb{R}, \qquad (73)$$

with ι equal to 1 or -1, as the case may be, and

[†]The continuity of $\overset{o}{\nu}$, $\hat{J}\chi$, and $\overset{o}{\upsilon}$ on $\partial\mathcal{R}\times\mathbb{R}$ (for $q \geq 1$) is sufficiently obvious.

$$\hat{J}\chi = \frac{|\overset{\circ}{N}|_3}{|(h^{-1})_{,1}\circ h\times(h^{-1})_{,2}\circ h|_3} \quad \text{in} \quad U\times\mathbb{R}. \tag{74}$$

Now it is evident that $D_4^k\overset{\circ}{\upsilon}$ and $D_4^k\hat{J}\chi$ exist and are continuous at (P,ζ) for $k = 1,\ldots,q-1$, respectively, all k. Finally, since

$$\begin{aligned}\overset{\circ}{\upsilon}(P,\zeta) &:= \upsilon(X(P,\zeta),\zeta) = \nu^j(\chi(P,\zeta),\zeta)\cdot\chi^j_{,4}(P,\zeta)\\ &= \overset{\circ}{\nu}{}^j(P,\zeta)\cdot\chi^j_{,4}(P,\zeta) \qquad \text{for} \quad P\in\partial R, \quad \zeta\in\mathbb{R},\end{aligned} \tag{75}$$

we must have $D_4^k\overset{\circ}{\upsilon}\in C(\partial R\times\mathbb{R})$ for $k = 1,\ldots,q-1$, respectively, all k, recalling that $D_4^k\chi\in C(\partial R\times\mathbb{R};\mathbb{R}^3)$ for $k = 1,\ldots,q$, respectively, all k. This completes the proof of (iv). $\square$.

[V.48] R E M A R K S. (a) Let us agree to establish as standard the notations ν and υ introduced in [V.47.ii].

In the following, $M\in\mathbb{M}(q)$ for some $q\in\mathbb{N}\cup\{\infty\}$, and (R,χ) is a reference pair for M as in [V.45].

(b) In each of [V.47.v and vi], we have required that $(X,t)\in\mathbb{B}^{\circ}\cup\Omega^{\sigma}$, *i.e.*, $(X,t)\notin\partial\mathbb{B}$, which is in general necessary for the truth of those assertions. Indeed, suppose that $(X,t)\in\partial\mathbb{B}$, so that $X\in\partial B_t$, and $X = \chi(P_X,t)$ for some $P_X\in\partial R$. Let (U,h) be a coordinate system in ∂R with $P_X\in U$. Then the partial derivatives of $\tau(h^{-1}(\cdot);X,t)$: $h(U)\to\mathbb{R}$ will in general fail to exist at $h(P_X)$, so we cannot assert in this case that τ is in one of the classes $C^k(\partial R)$. The source of the difficulty here is that $\tau(P_X;X,t) = 0$, so $X = [\chi]_{(X,t)}(P_X)$, and grad r_X fails to exist at $[\chi]_{(X,t)}(P_X)$.

Similarly, in this case we can generally say neither that $[\chi]_{(X,t)}$ is a k-imbedding (although it is a homeomorphism of ∂R onto $\partial\mathbb{B}(X,t)$) nor that $\partial\mathbb{B}(X,t)$ is a (2,3;k)-manifold for any k; exceptions may occur if $\upsilon(X,t) = 0$ (and obviously do occur if M is null). Indeed, a few rough computations in this case $(X,t) \in \partial\mathbb{B}$ indicate that if $\upsilon(X,t) \neq 0$, $\partial\mathbb{B}(X,t)$ has at X a conoidal type of singularity, the severity of which increases with $|\upsilon(X,t)|$, while the sign of $\upsilon(X,t)$ determines whether the singularity "points into" or "out of" $\mathbb{B}(X,t)^{o}$; if $\upsilon(X,t) = 0$, there is no singularity present. We supply no details to support these rather vague statements, since we shall have no occasion to deal with the properties of $\partial\mathbb{B}(X,t)$ for $(X,t) \in \partial\mathbb{B}$.

(c) The bound

$$|\upsilon| \leq c^* \quad \text{on} \quad \partial\mathbb{B}, \tag{1}$$

following directly from the definition (V.47.2) and the inequality (V.46.3), has already been cited. Since $c^* < c$, the explicit representation (V.47.3) for $\nu_{\partial\mathbb{B}}$ readily yields, with (1),

$$(\nu^4_{\partial\mathbb{B}})^2 \leq (c^*)^2/(1+\upsilon^2) = (c^*)^2\nu^i_{\partial\mathbb{B}}\nu^i_{\partial\mathbb{B}} < c^2\nu^i_{\partial\mathbb{B}}\nu^i_{\partial\mathbb{B}}. \tag{2}$$

Inequality (2) says that $\partial\mathbb{B}$ is *time-like* with respect to either Maxwell's equations or the wave equation, in the usual sense.

(d) Each of the functions

$$\zeta \mapsto \operatorname{diam} B_\zeta \tag{3}$$

and

$$\zeta \mapsto \lambda_{\partial\mathcal{B}_\zeta}(\partial\mathcal{B}_\zeta) \tag{4}$$

is continuous on $\mathbb{R}$. In fact, we have, for each $\zeta \in \mathbb{R}$,

$$\begin{aligned} \operatorname{diam} \mathcal{B}_\zeta = \operatorname{diam} \partial\mathcal{B}_\zeta &= \sup \{|X(P,\zeta)-X(Q,\zeta)|_3| \quad P,Q \in \partial R\} \\ &= |X(P_\zeta,\zeta)-X(Q_\zeta,\zeta)|_3 \end{aligned}$$

for some $P_\zeta, Q_\zeta \in \partial R$, by the continuity of $X(\cdot,\zeta)$ and the compactness of ∂R. Thus, for any $\xi \in \mathbb{R}$,

$$\begin{aligned} \operatorname{diam} \mathcal{B}_\zeta - \operatorname{diam} \mathcal{B}_\xi &\leq |X(P_\zeta,\zeta)-X(Q_\zeta,\zeta)|_3 - |X(P_\zeta,\xi)-X(Q_\zeta,\xi)|_3 \\ &\leq |X(P_\zeta,\zeta)-X(P_\zeta,\xi)|_3 + |X(Q_\zeta,\zeta)-X(Q_\zeta,\xi)|_3, \end{aligned}$$

a similar inequality holding with the roles of ξ and ζ interchanged. Coupling these observations with the uniform continuity of $X|\ \partial R \times K$, for any compact $K \subset \mathbb{R}$, we can obviously conclude that (3) is continuous on $\mathbb{R}$ (even in the absence of a smoothness condition on M). Further, again for each $\zeta \in \mathbb{R}$,

$$\lambda_{\partial\mathcal{B}_\zeta}(\partial\mathcal{B}_\zeta) = \int_{\partial\mathcal{B}_\zeta} d\lambda_{\partial\mathcal{B}_\zeta} = \int_{\partial R} \hat{J}X(\cdot,\zeta)\, d\lambda_{\partial R},$$

whence the continuity of (4) results from that of $\hat{J}X$ on $\partial R \times \mathbb{R}$. Thus, in particular, it is legitimate to speak of the (finite) numbers

$$\max_{\zeta \in K} \operatorname{diam} \mathcal{B}_\zeta \quad \text{and} \quad \max_{\zeta \in K} \lambda_{\partial\mathcal{B}_\zeta}(\partial\mathcal{B}_\zeta),$$

for any compact $K \subset \mathbb{R}$.

(e) Fix $(X,t) \in \mathbb{B}^{\circ} \cup \Omega^{\sigma}$; we consider here the map $[X^*]_{(X,t)}$, proven in [V.41] to be a homeomorphism of ∂R onto $\partial\mathbb{B} \cap C\_(X,t)$ (for any $(X,t) \in \mathbb{R}^4$). Now, since

$$[X^*]_{(X,t)} = ([X]_{(X,t)}, t-\tau(\cdot;X,t)),$$

it is evident from [V.47.v and vi.1] that $[X^*]_{(X,t)} \in C^q(\partial R;\mathbb{R}^4)$. Moreover, a somewhat lengthy computation, of the sort carried out in the proof of [V.47.vi.5], produces the representation

$$\begin{aligned}
J[X^*]_{(X,t)} &= \{1-\tau_{;4}(\cdot;X,t)\}\cdot[\hat{J}X]_{(X,t)} \\
&\quad \cdot \Big| [\overset{\circ}{\upsilon}]_{(X,t)} + [\overset{\circ}{\upsilon}{}^{c}]_{(X,t)} \cdot \text{grad } r_X \circ [X]_{(X,t)} \\
&\quad + \frac{1}{c^2} [\overset{\circ}{\upsilon}]_{(X,t)} \times \text{grad } r_X \circ [X]_{(X,t)} \Big|_3 \\
&= \{1-\tau_{;4}(\cdot;X,t)\}\cdot[\hat{J}X]_{(X,t)} \\
&\quad \cdot \Big\{ \Big| [\overset{\circ}{\upsilon}]_{(X,t)} + [\overset{\circ}{\upsilon}{}^{c}]_{(X,t)} \cdot \text{grad } r_X \circ [X]_{(X,t)} \Big|_3^2 \\
&\quad + \frac{1}{c^2} \Big| [\overset{\circ}{\upsilon}]_{(X,t)} \times \text{grad } r_X \circ [X]_{(X,t)} \Big|_3^2 \Big\}^{1/2} \quad \text{on} \quad \partial R,
\end{aligned} \tag{5}$$

whence it is easy to see that

$$J[X^*]_{(X,t)} > 0 \quad \text{on} \quad \partial R.$$

Combining these facts, we see that $[X^*]_{(X,t)}$ is a q-imbedding, implying that $\partial\mathbb{B} \cap C\_(X,t)$ is a $(2,4;q)$-manifold and, for each $P \in \partial R$, $D[X^*]_{(X,t)}(P)\colon\ T_{\partial R}(P) \to T_{\partial\mathbb{B}\cap C\_(X,t)}([X^*]_{(X,t)}(P))$ is a bijection.

For the proofs of certain uniqueness theorems in Chapter 4 of Part I, we have need of a variant of the approximating-domains result of [I.2.43] which is tailored to the geometry associated with a motion. The appropriate statement is proven in [V.50], following the preparation provided in [V.49].

[V.49] L E M M A. *Let* $M \in \mathbb{M}(1)$, *and suppose that* α *and* $\beta \in \mathbb{R}$, *with* $\alpha < \beta$. *Then there exists a* $\delta = \delta(M,\alpha,\beta) > 0$ *such that whenever* $(Z,\zeta) \in (\partial\mathbb{B})_{[\alpha,\beta]}$, *i.e., whenever* $\zeta \in [\alpha,\beta]$ *and* $Z \in \partial\mathcal{B}_\zeta$, *then*

$$Z+s\cdot\nu(Z,\zeta) \in \mathcal{B}_\zeta' \quad \textit{if} \quad 0 < s < \delta,$$

and

$$Z+s\cdot\nu(Z,\zeta) \in \mathcal{B}_\zeta^o \quad \textit{if} \quad -\delta < s < 0.$$

P R O O F. (The reasoning is quite similar to that of [VI.59].) Fix $\zeta \in \mathbb{R}$, then $Z \in \partial\mathcal{B}_\zeta$. $\mathbb{B}^o$ is a 1-regular domain ([V.47.iii.2]) and $(Z,\zeta) \in \partial\mathbb{B} = \partial\{\mathbb{B}^o\}$, so there exist an open neighborhood of (Z,ζ) in $\mathbb{R}^4$, $U_{(Z,\zeta)}$, and a function $\Phi_{(Z,\zeta)} \in C^1(U_{(Z,\zeta)})$ such that

$$\text{grad}\ \Phi_{(Z,\zeta)}(Y,s) \neq 0 \quad \text{for each} \quad (Y,s) \in U_{(Z,\zeta)}, \tag{1}$$

$$\partial\mathbb{B}\cap U_{(Z,\zeta)} = \{(Y,s) \in U_{(Z,\zeta)} \mid \ \Phi_{(Z,\zeta)}(Y,s) = 0\}, \tag{2}$$

and

$$\mathbb{B}^{o} \cap U_{(Z,\zeta)} = \{(Y,s) \in U_{(Z,\zeta)} \mid \Phi_{(Z,\zeta)}(Y,s) < 0\}; \tag{3}$$

obviously, then,

$$\Omega^{\sigma} \cap U_{(Z,\zeta)} = \mathbb{B}^{o-'} \cap U_{(Z,\zeta)} = \{(Y,s) \in U_{(Z,\zeta)} \mid \Phi_{(Z,\zeta)}(Y,s) > 0\}. \tag{4}$$

It follows that $\text{grad}\ \Phi_{(Z,\zeta)}(Y,s)$ is an exterior normal for $\partial\mathbb{B}$ at each $(Y,s) \in \partial\mathbb{B} \cap U_{(Z,\zeta)}$ (cf., [I.2.32.b]). Recalling the form of the exterior unit normal given in (V.47.3), we must then have

$$\frac{\text{grad}\ \Phi_{(Z,\zeta)}(Y,s)}{|\text{grad}\ \Phi_{(Z,\zeta)}(Y,s)|_n} = \frac{1}{\sqrt{\{1+\upsilon^2(Y,s)\}}} \cdot \{\upsilon^i(Y,s)e_i^{(4)} - \upsilon(Y,s)e_4^{(4)}\},$$
$$(Y,s) \in \partial\mathbb{B} \cap U_{(Z,\zeta)}. \tag{5}$$

Define $\text{grad}^{\wedge}\ \Phi_{(Z,\zeta)}\colon\ U_{(Z,\zeta)} \to \mathbb{R}^3$ by

$$\text{grad}^{\wedge}\ \Phi_{(Z,\zeta)} := (\Phi_{(Z,\zeta),1}, \Phi_{(Z,\zeta),2}, \Phi_{(Z,\zeta),3}); \tag{6}$$

from (5), it is clear that $\text{grad}^{\wedge}\ \Phi_{(Z,\zeta)}(Y,s)$ is an exterior normal for $\partial\mathcal{B}_s = \partial\{\mathcal{B}_s^o\}$ at Y, for each $(Y,s) \in \partial\mathbb{B} \cap U_{(Z,\zeta)}$. In particular, it can vanish at no point of the latter set, which includes (Z,ζ). Using the obvious continuity of $\text{grad}^{\wedge}\ \Phi_{(Z,\zeta)}$ on $U_{(Z,\zeta)}$, we can therefore choose a positive $\varepsilon_{(Z,\zeta)}$ for which we have $B^4_{2\varepsilon_{(Z,\zeta)}}(Z,\zeta) \subset U_{(Z,\zeta)}$ and $|\text{grad}^{\wedge}\ \Phi_{(Z,\zeta)}(Y,s)|_3 > 0$ for each $(Y,s) \in B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^-$. Setting

$$M_{(Z,\zeta)} := \sup\{|\text{grad}^{\wedge}\ \Phi_{(Z,\zeta)}(Y,s)|_3 \mid (Y,s) \in B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^-\},$$

and

$$m_{(Z,\zeta)} := \inf \{ |\mathrm{grad}^\wedge \Phi_{(Z,\zeta)}(Y,s)|_3| \quad (Y,s) \in B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^- \},$$

it is certainly true that $M_{(Z,\zeta)} < \infty$ and $m_{(Z,\zeta)} > 0$.

Now, whenever $(Y,s) \in B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^-$ and $|\sigma| < \tilde{\varepsilon}_{(Z,\zeta)} := \varepsilon_{(Z,\zeta)}/M_{(Z,\zeta)}$, it is quite easy to see that

$$(Y+\sigma\cdot \mathrm{grad}^\wedge \Phi_{(Z,\zeta)}(Y,s),s) \in B^4_{2\varepsilon_{(Z,\zeta)}}(Z,\zeta) \subset U_{(Z,\zeta)}.$$

This allows us to define $\psi_{(Y,s)}: (-\tilde{\varepsilon}_{(Z,\zeta)},\tilde{\varepsilon}_{(Z,\zeta)}) \to \mathbb{R}$ *via*

$$\psi_{(Y,s)}(\sigma) := \Phi_{(Z,\zeta)}(Y+\sigma\cdot \mathrm{grad}^\wedge \Phi_{(Z,\zeta)}(Y,s),s) \quad \text{for} \quad |\sigma| < \tilde{\varepsilon}_{(Z,\zeta)},$$
$$\text{for each} \quad (Y,s) \in B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^-; \tag{7}$$

then $\psi_{(Y,s)} \in C^1(-\tilde{\varepsilon}_{(Z,\zeta)},\tilde{\varepsilon}_{(Z,\zeta)})$ for each such (Y,s). We quickly compute

$$\psi_{(Y,s)}(0) = 0 \quad \text{if} \quad (Y,s) \in B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^- \cap \partial B \tag{8}$$

$$\psi'_{(Y,s)}(\sigma) = \mathrm{grad}^\wedge \Phi_{(Z,\zeta)}(Y+\sigma\cdot \mathrm{grad}^\wedge \Phi_{(Z,\zeta)}(Y,s),s) \bullet \mathrm{grad}^\wedge \Phi_{(Z,\zeta)}(Y,s)$$
$$\text{for} \quad |\sigma| < \tilde{\varepsilon}_{(Z,\zeta)}, \quad (Y,s) \in B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^-, \tag{9}$$

and

$$\psi'_{(Y,s)}(0) = |\mathrm{grad}^\wedge \Phi_{(Z,\zeta)}(Y,s)|_3^2 \geq m^2_{(Z,\zeta)}$$
$$\text{for} \quad (Y,s) \in B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^-. \tag{10}$$

Directly from (9), the map $(Y,s,\sigma) \mapsto \psi'_{(Y,s)}(\sigma)$ is continuous, and so also uniformly continuous, on the compact set $B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^- \times \left[-\frac{1}{2}\tilde{\varepsilon}_{(Z,\zeta)}, \frac{1}{2}\tilde{\varepsilon}_{(Z,\zeta)}\right]$ and satisfies (10) on $B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^- \times \{0\}$. From these facts, one can easily show that there exists a number $\delta_{(Z,\zeta)} \in (0, \frac{1}{2}\tilde{\varepsilon}_{(Z,\zeta)}]$ for which

$$\psi'_{(Y,s)}(\sigma) > \frac{1}{2} m_{(Z,\zeta)} > 0 \quad \text{for} \quad (Y,s) \in B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)^-, \quad |\sigma| < \delta_{(Z,\zeta)}. \tag{11}$$

Select any $(Y,s) \in \partial\mathbb{B} \cap B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)$ and any $\sigma \in (0, \delta_{(Z,\zeta)})$: using the mean-value theorem along with (8) and (11), for some $\tilde{\sigma} \in (0,\sigma)$ we can write

$$\begin{aligned}\Phi_{(Z,\zeta)}(Y+\sigma\cdot \text{grad}^\wedge\ \Phi_{(Z,\zeta)}(Y,s),s) &= \psi_{(Y,s)}(\sigma)\\ &= \psi_{(Y,s)}(0)+\psi'_{(Y,s)}(\tilde{\sigma})\cdot\sigma\\ &= \psi'_{(Y,s)}(\tilde{\sigma})\cdot\sigma\\ &> 0\ ;\end{aligned}$$

having already seen that $(Y+\sigma\cdot \text{grad}^\wedge\ \Phi_{(Z,\zeta)}(Y,s),s) \in U_{(Z,\zeta)}$, and recalling (4), we deduce that

$$(Y+\sigma\cdot \text{grad}^\wedge\ \Phi_{(Z,\zeta)}(Y,s),s) \in \Omega^\sigma.$$

This implies, in turn, that

$$Y+\sigma\cdot \operatorname{grad}^{\wedge} \Phi_{(Z,\zeta)}(Y,s) \in \mathcal{B}'_s \quad \text{whenever} \quad (Y,s) \in \partial\mathbb{B}\cap B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta)$$

$$\text{and} \quad \sigma \in (0,\delta_{(Z,\zeta)}).$$

Thus, since $\nu = |\operatorname{grad}^{\wedge} \Phi_{(Z,\zeta)}|_3^{-1}\cdot \operatorname{grad}^{\wedge} \Phi_{(Z,\zeta)}$ on $\partial\mathbb{B}\cap U_{(Z,\zeta)}$,

$$Y+\sigma\nu(Y,s) \in \mathcal{B}'_s \quad \text{whenever} \quad (Y,s) \in \partial\mathbb{B}\cap B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta) \quad \text{and} \quad \sigma \in (0,m_{(Z,\zeta)}\cdot\delta_{(Z,\zeta)}). \tag{12}$$

It can be shown in a similar fashion that

$$Y+\sigma\nu(Y,s) \in \mathcal{B}^o_s \quad \text{whenever} \quad (Y,s) \in \partial\mathbb{B}\cap B^4_{\varepsilon_{(Z,\zeta)}}(Z,\zeta) \quad \text{and} \quad \sigma \in (-m_{(Z,\zeta)}\cdot\delta_{(Z,\zeta)},0). \tag{13}$$

Having (12) and (13), which hold for each $(Z,\zeta) \in \partial\mathbb{B}$, the proof of the lemma can easily be completed: since $(\partial\mathbb{B})_{[\alpha,\beta]}$ is compact, there can be found a finite set $\{(Z_i,\zeta_i)\}_{i=1}^{\ell}$ in $(\partial\mathbb{B})_{[\alpha,\beta]}$ such that $\{B^4_{\varepsilon_{(Z_i,\zeta_i)}}(Z_i,\zeta_i)\}_{i=1}^{\ell}$ covers $(\partial\mathbb{B})_{[\alpha,\beta]}$. By setting

$$\delta(M,\alpha,\beta) := \min\, \{m_{(Z_i,\zeta_i)}\delta_{(Z_i,\zeta_i)}\}_{i=1}^{\ell},$$

one can readily check that we obtain a number possessing the required properties. □.

We proceed to the desired Lemma [V.50]. See Figure 8.

[V.50] L E M M A. *Let* M *be a motion in* $\mathbb{M}(q)$, *for some* $q \geq 2$.

Choose α *and* $\beta \in \mathbb{R}$, *with* $\alpha < \beta$. *Define, for each* $\varepsilon \in \mathbb{R}$, $G^{\varepsilon}_{(\alpha,\beta)}\colon\ (\partial\mathbb{B})_{(\alpha,\beta)} \to \mathbb{R}^4$ *by*

$$G^{\varepsilon}_{(\alpha,\beta)}(Z,\zeta) := (Z,\zeta)+\varepsilon(\nu(Z,\zeta),0) \qquad \text{for} \qquad (Z,\zeta) \in (\partial\mathbb{B})_{(\alpha,\beta)},$$

and set

$$\mathbb{B}^{\varepsilon}_{(\alpha,\beta)} := (\mathbb{B}^{o})^{\varepsilon}_{(\alpha,\beta)} := \cup_{\alpha<\zeta<\beta} \ \{\{Y \in B^{o}_{\zeta} |\ \text{dist}\ (Y,\partial B_{\zeta}) > -\varepsilon\}\times\{\zeta\}\},$$

$$\text{for} \qquad \varepsilon < 0,$$

$$\mathbb{B}^{\varepsilon}_{(\alpha,\beta)} := (\Omega^{\sigma})^{\varepsilon}_{(\alpha,\beta)} := \cup_{\alpha<\zeta<\beta} \ \{\{Y \in B'_{\zeta} |\ \text{dist}\ (Y,\partial B_{\zeta}) > \varepsilon\}\times\{\zeta\}\},$$

$$\text{for} \qquad \varepsilon > 0,$$

and

$$S^{\varepsilon}_{(\alpha,\beta)} := \partial\{\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}\}\cap\{\mathbb{R}^3\times(\alpha,\beta)\}.$$

Then there exists a positive ε_0, *depending upon* M, α, *and* β, *such that, for* $0 < |\varepsilon| < \varepsilon_0$,

(*i*) $G^{\varepsilon}_{(\alpha,\beta)}$ *is a* $(q-1)$*-imbedding, taking* $(\partial\mathbb{B})_{(\alpha,\beta)}$ *onto* $S^{\varepsilon}_{(\alpha,\beta)}$;

(*ii*) *if* $\varepsilon < 0$, $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ *is a normal domain;*

(*iii*) *if* $\varepsilon > 0$, $\tilde{c} > c^*$, *and* $\rho > 0$ *with* $B^3_{\tilde{c}(\beta-\alpha)}(B_\alpha) \subset B^3_{\rho/2}(0)$, *then the set*

$$\mathbb{B}^{\varepsilon,\rho}_{(\alpha,\beta)} := \mathbb{B}^{\varepsilon}_{(\alpha,\beta)}\cap\{B^3_{\rho}(0)\times(\alpha,\beta)\}$$

is a normal domain;

and

(*iv*) *the exterior unit normal field for* $S^{\varepsilon}_{(\alpha,\beta)} \subset \partial\{\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}\}$ *is given by*

$$\nu^{\varepsilon}_{(\alpha,\beta)} := \nu_{S^{\varepsilon}_{(\alpha,\beta)}} = -\operatorname{sgn}\varepsilon \cdot \nu_{\partial\mathbb{B}} \circ (G^{\varepsilon}_{(\alpha,\beta)})^{-1}.$$

Further,

(*v*) $\lim_{\varepsilon \to 0} JG^{\varepsilon}_{(\alpha,\beta)} = 1$ *uniformly on* $(\partial\mathbb{B})_{(\alpha,\beta)}$.

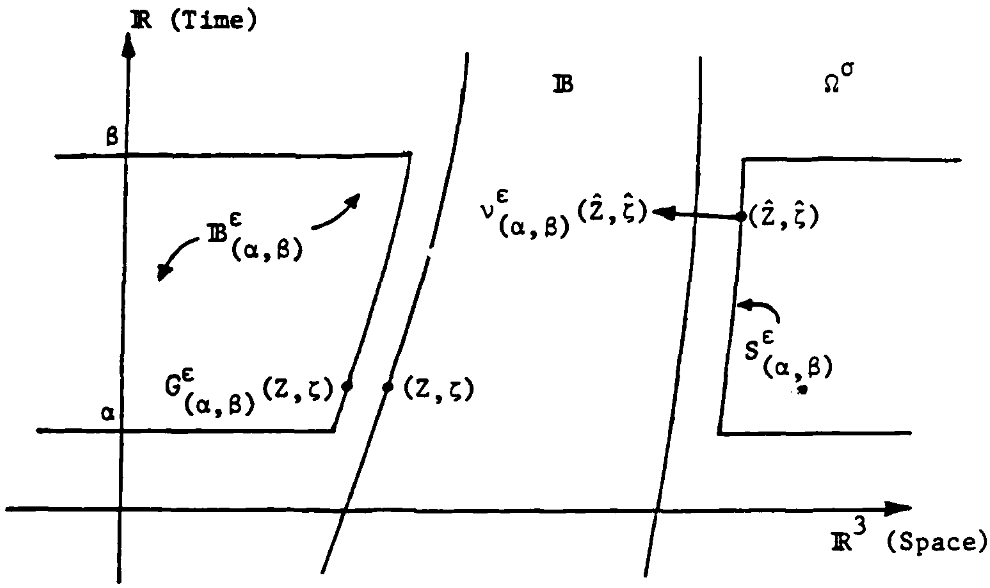
P R O O F. The arguments here are, in many respects, similar to those adduced in the proof of [I.2.43]; cf., [VI.68]. We shall first make various simple observations: since $q \geq 2$, we have (at least) the inclusion $\nu \in C^1(\partial\mathbb{B};\mathbb{R}^3)$, so that $\nu|K$ is Lipschitz continuous whenever K is a compact subset of $\partial\mathbb{B}$. In particular, this is true with $K = (\partial\mathbb{B})_{[\alpha,\beta]}$, so that there exists a positive a, depending on α, β, for which

$$|\nu(Z_2,\zeta_2)-\nu(Z_1,\zeta_1)|_3 \leq a|(Z_2,\zeta_2)-(Z_1,\zeta_1)|_4$$

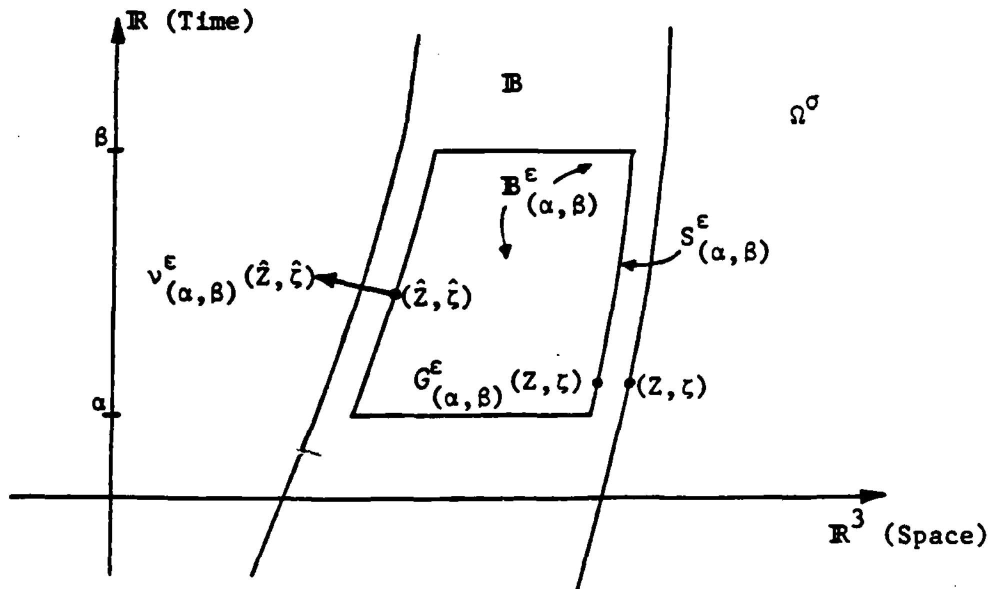
$$\text{whenever} \quad (Z_1,\zeta_1) \quad \text{and} \quad (Z_2,\zeta_2) \in (\partial\mathbb{B})_{[\alpha,\beta]}.$$

Surely, then, $\{\mathcal{B}^{o}_{\zeta}\}_{\alpha\leq\zeta\leq\beta}$ is a uniformly Lyapunov family, for which a set of uniform Lyapunov constants is $(a,1,d)$, provided $d \in (0, \frac{1}{2a})$. Note also that $(\partial\mathbb{B})_{(\alpha,\beta)}$,

$$(\partial\mathbb{B})_{(\alpha,\beta)} := \partial\mathbb{B}\cap\{\mathbb{R}^3\times(\alpha,\beta)\},$$



(b) $\varepsilon > 0$



(a) $\varepsilon < 0$

FIGURE 8. Approximating domains of Lemma [V.50]

being a relatively open subset of the $(3,4;q)$-manifold $\partial\mathbb{B}$, is again a $(3,4;q)$-manifold. For each $\zeta \in \mathbb{R}$, $(\partial\mathbb{B})_\zeta := \partial\mathcal{B}_\zeta \times \{\zeta\}$ is easily seen to be a $(2,4;q)$-manifold. It is convenient to define, for each $\varepsilon \in \mathbb{R}$, $G^\varepsilon\colon\ \partial\mathbb{B} \to \mathbb{R}^4$ according to

$$G^\varepsilon(Z,\zeta) := (Z,\zeta)+\varepsilon(\nu(Z,\zeta),0) \qquad \text{for each} \qquad (Z,\zeta) \in \partial\mathbb{B}$$

(so that $G^\varepsilon_{(\alpha,\beta)} = G^\varepsilon|\ (\partial\mathbb{B})_{(\alpha,\beta)}$), and write

$$G^\varepsilon_{[\alpha,\beta]} := G^\varepsilon|\ (\partial\mathbb{B})_{[\alpha,\beta]}.$$

Observe that, for any $\zeta \in \mathbb{R}$ and non-zero ε with $|\varepsilon|$ sufficiently small (depending upon ζ), we can regard [I.2.43] as giving a description of $G^\varepsilon(\cdot,\zeta)\colon\ \partial\mathcal{B}_\zeta \to \mathbb{R}^4$; *e.g.*, this map is a $(q-1)$-imbedding and, if, say, $\varepsilon < 0$, takes $\partial\mathcal{B}_\zeta$ onto the set $\{Y \in \mathcal{B}^o_\zeta|\ \text{dist}\ (Y,\partial\mathcal{B}_\zeta) = -\varepsilon\}\times\{\zeta\}$, which must then be a $(2,4;q-1)$-manifold. Finally, by [V.49], there exists a positive $\delta^* = \delta^*(M,\alpha,\beta)$ such that

$$\left.\begin{array}{ll} Z+\sigma\nu(Z,\zeta) \in \mathcal{B}'_\zeta & \text{if} \quad 0 < \sigma < \delta^*, \\ Z+\sigma\nu(Z,\zeta) \in \mathcal{B}^o_\zeta & \text{if} \quad -\delta^* < \sigma < 0 \end{array}\right\} \text{for each} \quad (Z,\zeta) \in (\partial\mathbb{B})_{[\alpha,\beta]}. \quad (1)$$

(i) Let (U,h) be a coordinate system in $\partial\mathbb{B}$. For any $\varepsilon \in \mathbb{R}$,

$$G^\varepsilon \circ h^{-1} = h^{-1}+\varepsilon(\nu\circ h^{-1},0) \qquad \text{on} \qquad h(U) \subset \mathbb{R}^3;$$

since $\nu \in C^{q-1}(\partial\mathbb{B};\mathbb{R}^3)$, it is readily apparent that $G^\varepsilon\circ h^{-1} \in C^{q-1}(h(U);\mathbb{R}^4)$. Consequently, $G^\varepsilon \in C^{q-1}(\partial\mathbb{B};\mathbb{R}^4)$. Certainly, G^ε

is then continuous. Any coordinate system in $(\partial\mathbb{B})_{(\alpha,\beta)}$ is also a coordinate system in $\partial\mathbb{B}$, while $G^{\varepsilon}_{(\alpha,\beta)} = G^{\varepsilon}|\ (\partial\mathbb{B})_{(\alpha,\beta)}$, so we can conclude as well that $G^{\varepsilon}_{(\alpha,\beta)} \in C^{q-1}((\partial\mathbb{B})_{(\alpha,\beta)};\mathbb{R}^4)$, and $G^{\varepsilon}_{(\alpha,\beta)} \in C((\partial\mathbb{B})_{(\alpha,\beta)};\mathbb{R}^4)$, for each real ε.

We can show that $G^{\varepsilon}_{[\alpha,\beta]}$ is injective if $|\varepsilon| < 1/a$. Indeed, if (Z_1,ζ_1), $(Z_2,\zeta_2) \in (\partial\mathbb{B})_{[\alpha,\beta]}$ with $G^{\varepsilon}_{[\alpha,\beta]}(Z_1,\zeta_1) = G^{\varepsilon}_{[\alpha,\beta]}(Z_2,\zeta_2)$, then $\zeta_1 = \zeta_2$ and $Z_1+\varepsilon\nu(Z_1,\zeta_1) = Z_2+\varepsilon\nu(Z_2,\zeta_2)$, so

$$|Z_2-Z_1|_3 = |\varepsilon|\cdot|\nu(Z_2,\zeta_1)-\nu(Z_1,\zeta_1)|_3 \leq a\cdot|\varepsilon|\cdot|Z_2-Z_1|_3.$$

Thus, if $a\cdot|\varepsilon| < 1$, it is clear that $Z_1 = Z_2$, proving our claim. But now we can use the compactness of $(\partial\mathbb{B})_{[\alpha,\beta]}$ to assert that $G^{\varepsilon}_{[\alpha,\beta]}\colon\ (\partial\mathbb{B})_{[\alpha,\beta]} \to G^{\varepsilon}((\partial\mathbb{B})_{[\alpha,\beta]})$ is a homeomorphism if $|\varepsilon| < 1/a$, whence its restriction $G^{\varepsilon}_{(\alpha,\beta)}\colon\ (\partial\mathbb{B})_{(\alpha,\beta)} \to G^{\varepsilon}((\partial\mathbb{B})_{(\alpha,\beta)})$ must also be a homeomorphism for these same ε.

Now, to prove that $G^{\varepsilon}_{(\alpha,\beta)}$ is a $(q-1)$-imbedding for $|\varepsilon|$ sufficiently small, we must prove that $JG^{\varepsilon}_{(\alpha,\beta)}$ is positive on $(\partial\mathbb{B})_{(\alpha,\beta)}$ for such ε. For this, it is clearly sufficient to show that

$$\lim_{\varepsilon\to 0}\ (JG^{\varepsilon})|\ (\partial\mathbb{B})_{[\alpha,\beta]} = 1 \quad \text{uniformly on} \quad (\partial\mathbb{B})_{[\alpha,\beta]}, \tag{2}$$

since $JG^{\varepsilon}_{(\alpha,\beta)} = (JG^{\varepsilon})|\ (\partial\mathbb{B})_{(\alpha,\beta)}$. Moreover, statement (v) shall follow immediately, once (2) has been verified. To prove (2), we use the compactness of $(\partial\mathbb{B})_{[\alpha,\beta]}$ and the familiar properties of coordinate systems to construct a family $\{(U_\iota,h_\iota)\}_{\iota\in I}$ of coordinate systems in $\partial\mathbb{B}$ such that I is finite and $\{U_\iota\}_{\iota\in I}$ covers $(\partial\mathbb{B})_{[\alpha,\beta]}$, while, for each $\iota \in I$, all partial derivatives of h_ι^{-1} and $\nu o h_\iota^{-1}$ are bounded on $h_\iota(U_\iota)$, and $|\bigwedge_{i=1}^{3} (h_\iota^{-1})_{,i}|$ is bounded below by a positive number on $h_\iota(U_\iota)$. Since

$$(G^\varepsilon o h_\iota^{-1})_{,i} = \{(h_\iota^{-1})^j_{,i}+\varepsilon\cdot(\nu^j o h_\iota^{-1})_{,i}\}e_j^{(4)}+(h_\iota^{-1})^4_{,i}e_4^{(4)} \quad \text{on} \quad h_\iota(U_\iota),$$

a short computation shows that for, say, $|\varepsilon| < 1$,

$$|\bigwedge_{i=1}^{3} (G^\varepsilon o h_\iota^{-1})_{,i}| \leq |\bigwedge_{i=1}^{3} (h_\iota^{-1})_{,i}|+M_\iota\cdot|\varepsilon| \quad \text{on} \quad h_\iota(U_\iota)$$

for each $\iota \in I$, for certain positive numbers $\{M_\iota\}_{\iota\in I}$. Thus, in view of (I.2.11.2),

$$JG^\varepsilon| \ U_\iota \leq 1+M_\iota'\cdot|\varepsilon| \quad \text{for each} \quad \iota \in I, \tag{3}$$

for appropriate positive numbers $\{M_\iota'\}_{\iota\in I}$. But now (2) follows from (3), recalling that I is finite and $\{U_\iota\}_{\iota\in I}$ covers $(\partial\mathbb{B})_{[\alpha,\beta]}$. We have now shown that $G^\varepsilon_{(\alpha,\beta)}$ is a $(q-1)$-imbedding whenever $|\varepsilon|$ is sufficiently small. This implies, among other things, that $G^\varepsilon_{(\alpha,\beta)}((\partial\mathbb{B})_{(\alpha,\beta)})$ is a $(3,4;q-1)$-manifold for such ε.

For the proof of the second statement in (i), we shall first establish the equalities

$$G^\varepsilon_{(\alpha,\beta)}((\partial\mathbb{B})_{(\alpha,\beta)}) = \cup_{\alpha<\zeta<\beta} \ \{\{Y \in \mathcal{B}_\zeta'| \ \text{dist}\,(Y,\partial\mathcal{B}_\zeta) = \varepsilon\}\times\{\zeta\}\} \tag{4}$$

$$\text{if} \quad \varepsilon > 0,$$

$$G^{\varepsilon}_{(\alpha,\beta)}((\partial\mathbb{B})_{(\alpha,\beta)}) = \cup_{\alpha<\zeta<\beta} \{\{Y \in \mathcal{B}^{o}_{\zeta} | \ \text{dist}\ (Y,\partial\mathcal{B}_{\zeta}) = -\varepsilon\}\times\{\zeta\}\} \quad \text{if} \quad \varepsilon < 0, \tag{5}$$

whenever $|\varepsilon|$ is sufficiently small. Considering (4), we suppose that $0 < \varepsilon < \min\{\delta^*, d/2, 1/(2\hat{a})\}$, with $\hat{a}$ being given in terms of a in [I.2.37.iii.4]; note that $\hat{a} > a$. Select $\zeta \in (\alpha,\beta)$ and $Y \in \mathcal{B}'_{\zeta}$ with $\text{dist}\ (Y,\partial\mathcal{B}_{\zeta}) = \varepsilon$. Then ([I.2.20]) there exists $X_Y \in \partial\mathcal{B}_{\zeta}$ for which $|Y-X_Y|_3 = \varepsilon$ and $Y-X_Y \in N_{\partial\mathcal{B}_{\zeta}}(X_Y)$, whence Y must be given by one of $X_Y+\varepsilon\nu(X_Y,\zeta)$, $X_Y-\varepsilon\nu(X_Y,\zeta)$. But the latter lies in $\mathcal{B}^{o}_{\zeta}$, since $0 < \varepsilon < \delta^*$, so

$$(Y,\zeta) = (X_Y+\varepsilon\nu(X_Y,\zeta),\zeta) = G^{\varepsilon}_{(\alpha,\beta)}(X_Y,\zeta),$$

proving one of the inclusions required. For the other, again let $\zeta \in (\alpha,\beta)$ and choose $Z \in \partial\mathcal{B}_{\zeta}$, so that $(Z,\zeta) \in (\partial\mathbb{B})_{(\alpha,\beta)}$. Then

$$G^{\varepsilon}_{(\alpha,\beta)}(Z,\zeta) = (Z+\varepsilon\nu(Z,\zeta),\zeta) \in \mathcal{B}'_{\zeta}\times\{\zeta\},$$

since $0 < \varepsilon < \delta^*$. We must show that

$$\delta := \text{dist}\ (Z+\varepsilon\nu(Z,\zeta),\partial\mathcal{B}_{\zeta}) := \inf_{Y \in \partial\mathcal{B}_{\zeta}} |Z+\varepsilon\nu(Z,\zeta)-Y|_3 = \varepsilon.$$

The inequalities $0 < \delta \leq \varepsilon$ obviously hold. Assume that the strict inequality $\delta < \varepsilon$ is true: appealing once more to [I.2.20] and the fact that $0 < \varepsilon < \delta^*$, we know that there is some $\tilde{Z} \in \partial\mathcal{B}_{\zeta}$ for which

$$|Z+\varepsilon\nu(Z,\zeta)-\tilde{Z}|_3 = \delta$$

and

$$Z+\varepsilon\nu(Z,\zeta) = \tilde{Z}+\delta\nu(\tilde{Z},\zeta);$$

clearly, it must be the case that $Z \neq \tilde{Z}$, by our assumption that $\delta < \varepsilon$. Since

$$\begin{aligned}|Z-\tilde{Z}|_3 &= |\delta\nu(\tilde{Z},\zeta)-\varepsilon\nu(Z,\zeta)|_3 \\ &\leq (\varepsilon-\delta)+\varepsilon|\nu(\tilde{Z},\zeta)-\nu(Z,\zeta)|_3 \\ &\leq (\varepsilon-\delta)+a\varepsilon\cdot|Z-\tilde{Z}|_3 \\ &< (\varepsilon-\delta)+\frac{1}{2}|Z-\tilde{Z}|_3,\end{aligned}$$

we have

$$|Z-\tilde{Z}|_3 < 2(\varepsilon-\delta) < 2\varepsilon < d,$$

i.e., $Z \in \partial\mathcal{B}_\zeta \cap B^3_d(\tilde{Z})$, and we can apply the estimate [I.2.37.iii.4] to obtain

$$|\nu(Z,\zeta)\bullet \operatorname{grad} r_{\tilde{Z}}(Z)| < \hat{a}\cdot r_{\tilde{Z}}(Z),$$

whence

$$2\varepsilon\cdot r_{\tilde{Z}}(Z)\cdot\nu(Z,\zeta)\bullet \operatorname{grad} r_{\tilde{Z}}(Z) > -2\hat{a}\varepsilon r^2_{\tilde{Z}}(Z). \tag{6}$$

Further,

$$\delta^2 = |Z+\varepsilon\nu(Z,\zeta)-\tilde{Z}|^2_3 = r^2_{\tilde{Z}}(Z)+\varepsilon^2+2\varepsilon\nu(Z,\zeta)\bullet(Z-\tilde{Z}),$$

giving

$$2\varepsilon \cdot r_{\bar{Z}}(Z) \cdot \nu(Z,\zeta) \bullet \text{grad } r_{\bar{Z}}(Z) + r_{\bar{Z}}^2(Z) = \delta^2 - \varepsilon^2 .$$

Remembering that $2\hat{a}\varepsilon < 1$, and using (6), we therefore arrive at the impossibility

$$0 < (1-2\hat{a}\varepsilon) r_{\bar{Z}}^2(Z) < \delta^2 - \varepsilon^2 < 0.$$

Thus, $\delta = \varepsilon$. Now (4) has been proven for all sufficiently small positive ε. An analogous argument serves to demonstrate that (5) is correct if $-\min\{\delta^*, d/2, 1/(2\hat{a})\} < \varepsilon < 0$.

To show that $G^{\varepsilon}_{(\alpha,\beta)}((\partial \mathbb{B})_{(\alpha,\beta)}) = S^{\varepsilon}_{(\alpha,\beta)}$ whenever $|\varepsilon|$ is small enough and non-zero, our intention is to prove that, for such ε,

$$S^{\varepsilon}_{(\alpha,\beta)} = \cup_{\alpha<\zeta<\beta} \{\{Y \in B'_{\zeta} | \text{ dist } (Y, \partial B_{\zeta}) = \varepsilon\} \times \{\zeta\}\} \quad \text{if} \quad \varepsilon > 0, \qquad (7)$$

while

$$S^{\varepsilon}_{(\alpha,\beta)} = \cup_{\alpha<\zeta<\beta} \{\{Y \in B^{o}_{\zeta} | \text{ dist } (Y, \partial B_{\zeta}) = -\varepsilon\} \times \{\zeta\}\} \quad \text{if} \quad \varepsilon < 0, \qquad (8)$$

whence the desired result shall follow, with (4) and (5). With a few preliminary developments, (7) and (8) shall follow easily. Let us first show that $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ is an open set for each non-zero ε. Choose $\varepsilon > 0$, and set

$$\mathbb{B}^{\varepsilon} := \cup_{\zeta \in \mathbb{R}} \{\{Y \in B'_{\zeta} | \text{ dist } (Y, \partial B_{\zeta}) > \varepsilon\} \times \{\zeta\}\}; \qquad (9)$$

then $\mathbb{B}^{\varepsilon}$ is open. To see this, suppose that $(X,t) \in \mathbb{B}^{\varepsilon}$ and write

$$\delta_1 := \text{dist }(X,\partial\mathcal{B}_t) := \inf\{|Y-X|_3 \mid Y \in \partial\mathcal{B}_t\},$$

so that $\delta_1 > \varepsilon$. Let

$$\eta_1 := \frac{1}{2c}(\delta_1-\varepsilon).$$

We shall show that

$$B^3_{\frac{1}{2}(\delta_1-\varepsilon)}(X)\times(t-\eta_1,\ t+\eta_1) \subset \mathbb{B}^{\varepsilon}, \tag{10}$$

which will certainly imply that $\mathbb{B}^{\varepsilon}$ is open. To prove (10), let $Y \in \mathbb{R}^3$ with $|Y-X|_3 < \frac{1}{2}(\delta_1-\varepsilon)$. Then

$$B^3_{\varepsilon+\frac{1}{2}(\delta_1-\varepsilon)}(Y) \subset \mathcal{B}_t', \tag{11}$$

for, if $|\hat{Y}-Y|_3 < \varepsilon+\frac{1}{2}(\delta_1-\varepsilon)$, then whenever $Z \in \mathcal{B}_t$,

$$\begin{aligned}|Z-\hat{Y}|_3 &\geq |Z-X|_3-|X-Y|_3-|Y-\hat{Y}|_3\\ &> \delta_1-\frac{1}{2}(\delta_1-\varepsilon)-\varepsilon-\frac{1}{2}(\delta_1-\varepsilon)\\ &= 0,\end{aligned}$$

so $\hat{Y} \in \mathcal{B}_t'$ (here, we have used the fact that $\delta_1 := \text{dist }(X,\partial\mathcal{B}_t) = \text{dist }(X,\mathcal{B}_t)$, which can easily be checked, since $X \in \mathcal{B}_t'$). Now, suppose that $|t-\zeta| < \eta_1$, so, obviously, $|t-\zeta| < \frac{1}{c}\{\varepsilon+\frac{1}{2}(\delta_1-\varepsilon)\}$. Combining (11) with the result of [V.13.ii], we deduce that

$$B^3_{\varepsilon+\frac{1}{2}(\delta_1-\varepsilon)-c|t-\zeta|}(Y) \subset \mathcal{B}_\zeta',$$

which shows immediately that $Y \in \mathcal{B}_\zeta'$ and $\text{dist }(Y,\partial\mathcal{B}_\zeta) =$

dist $(Y,\mathcal{B}_\zeta) > \varepsilon$, since $c|t-\zeta| < \frac{1}{2}(\delta_1-\varepsilon)$, *i.e.*, $\varepsilon+\frac{1}{2}(\delta_1-\varepsilon)-c(t-\zeta) > \varepsilon$. Thus, (Y,ζ) lies in $\mathbb{B}^\varepsilon$, whence (10) is proven. But then, as remarked, it follows that $\mathbb{B}^\varepsilon$ is open. Now,

$$\mathbb{B}^\varepsilon_{(\alpha,\beta)} = \mathbb{B}^\varepsilon \cap \{\mathbb{R}^3 \times (\alpha,\beta)\},$$

so we can conclude that $\mathbb{B}^\varepsilon_{(\alpha,\beta)}$ is open for each $\varepsilon > 0$. Further, defining

$$\mathbb{B}^\varepsilon := \cup_{\zeta\in\mathbb{R}} \{\{Y \in \mathcal{B}^o_\zeta | \text{ dist } (Y,\partial\mathcal{B}_\zeta) > -\varepsilon\}\times\{\zeta\}\} \quad \text{for} \quad \varepsilon < 0,$$

one can show that $\mathbb{B}^\varepsilon$ is open for each $\varepsilon < 0$, using similar reasoning which appeals to [V.13.i] at the appropriate juncture (note that it may well be that $\mathbb{B}^\varepsilon = \emptyset$ if $|\varepsilon|$ is sufficiently large, but the same method of proof is valid). From this it clearly follows that $\mathbb{B}^\varepsilon_{(\alpha,\beta)}$ is open whenever $\varepsilon < 0$.

Next, defining

$$\tilde{\mathbb{B}}^\varepsilon := \cup_{\zeta\in\mathbb{R}} \{\{Y \in \mathcal{B}'_\zeta | \text{ dist } (Y,\partial\mathcal{B}_\zeta) \geq \varepsilon\}\times\{\zeta\}\} \quad \text{if} \quad \varepsilon > 0, \tag{13}$$

and

$$\tilde{\mathbb{B}}^\varepsilon := \cup_{\zeta\in\mathbb{R}} \{\{Y \in \mathcal{B}^o_\zeta | \text{ dist } (Y,\partial\mathcal{B}_\zeta) \geq -\varepsilon\}\times\{\zeta\}\} \quad \text{if} \quad \varepsilon < 0, \tag{14}$$

we shall show that each $\tilde{\mathbb{B}}^\varepsilon$ is a closed subset of $\mathbb{R}^4$. In fact, select any $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$. Let $((Y_n,\zeta_n))_{n=1}^\infty$ be a sequence in $\tilde{\mathbb{B}}^\varepsilon$, converging to $(Y_0,\zeta_0) \in \mathbb{R}^4$. If $\varepsilon > 0$, then $\{(Y_n,\zeta_n)\}_{n=1}^\infty \subset \Omega^\sigma$, so $(Y_0,\zeta_0) \in \Omega^{\sigma-}$, *i.e.*, $Y_0 \in B'^-_{\zeta_0} = \mathcal{B}'_{\zeta_0} \cup \partial\mathcal{B}_{\zeta_0}$, whereas if $\varepsilon < 0$, then $\{(Y_n,\zeta_n)\}_{n=1}^\infty \subset \mathbb{B}^o$, so $(Y_0,\zeta_0) \in \mathbb{B}$, *i.e.*,

$Y_0 \in B_{\zeta_0} = B^o_{\zeta_0} \cup \partial B_{\zeta_0}$. Therefore, if we show that

$$\text{dist}\,(Y_0, \partial B_{\zeta_0}) := \inf\{|Y-Y_0|_3 \mid Y \in \partial B_{\zeta_0}\} \geq |\varepsilon|, \tag{15}$$

it will be immediately evident that $Y_0 \in B'_{\zeta_0}$ if $\varepsilon > 0$, $Y_0 \in B^o_{\zeta_0}$ if $\varepsilon < 0$, and $(Y_0, \zeta_0) \in \tilde{\mathbb{B}}^\varepsilon$ in either case. To prove (15), observe first that

$$\lim_{n \to \infty} Y_n = Y_0, \qquad \lim_{n \to \infty} \zeta_n = \zeta_0, \tag{16}$$

and

$$\inf\{|Z-Y_n|_3 \mid Z \in \partial B_{\zeta_n}\} \geq |\varepsilon| \qquad \text{for each} \quad n \in \mathbb{N}. \tag{17}$$

Suppose, contrary to (15), that $\text{dist}\,(Y_0, \partial B_{\zeta_0}) < |\varepsilon|$; then there exists a point $\hat{Y}_0 \in \partial B_{\zeta_0}$ such that $|Y_0 - \hat{Y}_0|_3 = \text{dist}\,(Y_0, \partial B_{\zeta_0}) < |\varepsilon|$. Let us employ the reference pair $(B_{\zeta_0}, \chi^{\zeta_0})$ for M, wherein $\chi^{\zeta_0}(\cdot, \zeta_0)$ is the identity function on ∂B_{ζ_0}. By (16), we have, on the one hand,

$$\lim_{n \to \infty} |\chi^{\zeta_0}(\hat{Y}_0, \zeta_n) - Y_n|_3 = |\chi^{\zeta_0}(\hat{Y}_0, \zeta_0) - Y_0|_3 = |\hat{Y}_0 - Y_0|_3 < |\varepsilon|, \tag{18}$$

while, on the other, $\chi^{\zeta_0}(\hat{Y}_0, \zeta_n) \in \partial B_{\zeta_n}$, so, by (17),

$$|\chi^{\zeta_0}(\hat{Y}_0, \zeta_n) - Y_n|_3 \geq |\varepsilon| \qquad \text{for each} \quad n \in \mathbb{N}. \tag{19}$$

The contradiction resulting from (18) and (19) implies that (15) is correct. Thus, $(Y_0, \zeta_0) \in \tilde{\mathbb{B}}^\varepsilon$, and with this inclusion we can conclude that $\tilde{\mathbb{B}}^{\varepsilon}$ is closed. Now, if we write

$$\tilde{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]} := \tilde{\mathbb{B}}^{\varepsilon} \cap \{\mathbb{R}^3 \times [\alpha,\beta]\}$$

$$= \cup_{\alpha \leq \zeta \leq \beta} \{\{Y \in \mathcal{B}'_{\zeta} \mid \text{dist}(Y, \partial\mathcal{B}_{\zeta}) \geq \varepsilon\} \times \{\zeta\}\} \quad \text{if} \quad \varepsilon > 0, \tag{20}$$

and

$$\tilde{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]} := \tilde{\mathbb{B}}^{\varepsilon} \cap \{\mathbb{R}^3 \times [\alpha,\beta]\}$$

$$= \cup_{\alpha \leq \zeta \leq \beta} \{\{Y \in \mathcal{B}^{o}_{\zeta} \mid \text{dist}(Y, \partial\mathcal{B}_{\zeta}) \geq -\varepsilon\} \times \{\zeta\}\} \quad \text{if} \quad \varepsilon < 0, \tag{21}$$

it is evident that $\tilde{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]}$ is a closed subset of $\mathbb{R}^4$ for each non-zero $\varepsilon \in \mathbb{R}$.

As our final preparation for the proofs of (7) and (8), we shall prove that

$$\tilde{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]} = \mathbb{B}^{\varepsilon-}_{(\alpha,\beta)} \quad \text{whenever } |\varepsilon| \text{ is sufficiently small}, \quad \varepsilon \neq 0. \tag{22}$$

First, since $\tilde{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]}$ is closed and contains $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$, the inclusion $\mathbb{B}^{\varepsilon-}_{(\alpha,\beta)} \subset \tilde{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]}$ must hold for all real non-zero ε. For the reverse inclusion, we note that for some $\varepsilon^* = \varepsilon^*(M,\alpha,\beta) > 0$,

$$G^{\sigma}(\partial\mathcal{B}_{\zeta} \times \{\zeta\}) = \{Y \in \mathcal{B}'_{\zeta} \mid \text{dist}(Y, \partial\mathcal{B}_{\zeta}) = \sigma\} \times \{\zeta\} \quad \text{if} \quad 0 < \sigma < \varepsilon^*, \tag{23}$$

and

$$G^{\sigma}(\partial\mathcal{B}_{\zeta} \times \{\zeta\}) = \{Y \in \mathcal{B}^{o}_{\zeta} \mid \text{dist}(Y, \partial\mathcal{B}_{\zeta}) = -\sigma\} \times \{\zeta\} \quad \text{if} \quad -\varepsilon^* < \sigma < 0, \tag{24}$$

whenever $\zeta \in [\alpha,\beta]$; the proof of this statement is essentially the same as that of (4) and (5). We examine first the case in which

$\varepsilon > 0$, and suppose also that $\varepsilon < \varepsilon^*$. We choose any point $(X,t) \in \tilde{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]}$ and consider in turn each of the possible cases:

(a) Suppose that $t \in (\alpha,\beta)$. We have $X \in \mathcal{B}'_t$, with $\operatorname{dist}(X,\partial\mathcal{B}_t) \geq \varepsilon$; if strict inequality holds, then (X,t) lies in $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)} \subset \mathbb{B}^{\varepsilon-}_{(\alpha,\beta)}$, so we shall assume that $\operatorname{dist}(X,\partial\mathcal{B}_t) = \varepsilon$. Because (23) is valid, we can write $X = \tilde{X}+\varepsilon\nu(\tilde{X},t)$ for some $\tilde{X} \in \partial\mathcal{B}_t$. Select a sequence $(\varepsilon_n)_{n=1}^{\infty}$ in $(\varepsilon,\varepsilon^*)$ which converges to ε. Then, because of (23), $\tilde{X}+\varepsilon_n\nu(\tilde{X},t) \in \mathcal{B}'_t$ and $\operatorname{dist}(\tilde{X}+\varepsilon_n\nu(\tilde{X},t),\partial\mathcal{B}_t) = \varepsilon_n > \varepsilon$ for each $n \in \mathbb{N}$, *i.e.*, the sequence $((\tilde{X}+\varepsilon_n\nu(\tilde{X},t),t))_{n=1}^{\infty}$ lies in $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ and converges to (X,t), so the latter is in $\mathbb{B}^{\varepsilon-}_{(\alpha,\beta)}$.

(b) Suppose that $t = \alpha$. If $\operatorname{dist}(X,\partial\mathcal{B}_\alpha) > \varepsilon$, then (X,α) is in the open set $\mathbb{B}^{\varepsilon}$ (cf., (9)), which therefore contains an open "pillbox" $B^3_a(X)\times(\alpha-b,\ \alpha+b)$, with $0 < b < \beta-\alpha$. Thus, the entire line segment $\{X\}\times(\alpha,\alpha+b)$ lies in $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$, clearly implying that $(X,\alpha) \in \mathbb{B}^{\varepsilon-}_{(\alpha,\beta)}$. Consider the other case, *viz.*, that in which $\operatorname{dist}(X,\partial\mathcal{B}_\alpha) = \varepsilon$: now we can write $X = \tilde{X}+\varepsilon\nu(\tilde{X},\alpha)$ for some $\tilde{X} \in \partial\mathcal{B}_\alpha$. Employing the reference pair $(\mathcal{B}_\alpha,\chi^\alpha)$ for $\mathcal{M}$, wherein $\chi^\alpha(\cdot,\alpha)$ is the identity on $\partial\mathcal{B}_\alpha$, choose a sequence $(s_n)_{n=1}^{\infty}$ from (α,β) converging to α, a sequence $(\varepsilon_n)_{n=1}^{\infty}$ in $(\varepsilon,\varepsilon^*)$ converging to ε, and form the sequence $(\tilde{X}_n := \chi^\alpha(\tilde{X},s_n)+\varepsilon_n\nu(\chi^\alpha(\tilde{X},s_n),s_n))_{n=1}^{\infty}$, which converges to $\chi^\alpha(\tilde{X},\alpha)+\varepsilon\nu(\chi^\alpha(\tilde{X},\alpha),\alpha) = \tilde{X}+\varepsilon\nu(\tilde{X},\alpha) = X$. Again by (23), $\tilde{X}_n \in \mathcal{B}'_{s_n}$ and $\operatorname{dist}(\tilde{X}_n,\partial\mathcal{B}_{s_n}) = \varepsilon_n > \varepsilon$ for each $n \in \mathbb{N}$. Thus, the sequence $((\tilde{X}_n,s_n))_{n=1}^{\infty}$ lies in $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ and converges to (X,α), whence the latter is in $\mathbb{B}^{\varepsilon-}_{(\alpha,\beta)}$.

(c) Suppose that $t = \beta$. The treatment of this case follows the same lines as that of (b); we omit the details.

The equality $\tilde{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]} = \mathbb{B}^{\varepsilon-}_{(\alpha,\beta)}$ therefore holds if $0 < \varepsilon < \varepsilon^*$; its proof for $-\varepsilon^* < \varepsilon < 0$ can be carried out in a similar manner, using (24), so (22) is true.

We can now quickly prove (7) and (8). In fact, if $0 < |\varepsilon| < \varepsilon^*$, using (22) and the fact that $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ is open,

$$\partial\{\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}\} = \mathbb{B}^{\varepsilon-}_{(\alpha,\beta)} \cap \mathbb{B}^{\varepsilon'-}_{(\alpha,\beta)} = \tilde{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]} \cap \mathbb{B}^{\varepsilon'}_{(\alpha,\beta)},$$

so if, say, $\varepsilon > 0$,

$$\begin{aligned}\partial\{\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}\} = \cup_{\alpha<\zeta<\beta} &\{\{Y \in \mathcal{B}'_{\zeta} \mid \text{dist}\,(Y,\partial\mathcal{B}_{\zeta}) = \varepsilon\}\times\{\zeta\}\} \\ &\cup\{\{Y \in \mathcal{B}'_{\alpha} \mid \text{dist}\,(Y,\partial\mathcal{B}_{\alpha}) \geq \varepsilon\}\times\{\alpha\}\} \\ &\cup\{\{Y \in \mathcal{B}'_{\beta} \mid \text{dist}\,(Y,\partial\mathcal{B}_{\beta}) \geq \varepsilon\}\times\{\beta\}\}.\end{aligned}$$

Recalling the definition $S^{\varepsilon}_{(\alpha,\beta)} := \partial\{\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}\}\cap\{\mathbb{R}^3\times(\alpha,\beta)\}$, (7) results if $0 < \varepsilon < \varepsilon^*$. In the same way, we obtain (8) for $-\varepsilon^* < \varepsilon < 0$.

Having (4), (5), (7), and (8) if $0 < |\varepsilon| < \varepsilon^*$, the equality $G^{\varepsilon}_{(\alpha,\beta)}((\partial\mathbb{B})_{(\alpha,\beta)}) = S^{\varepsilon}_{(\alpha,\beta)}$ holds for these ε, and (i) has now been completely substantiated. Note, for example, that

$S^{\varepsilon}_{(\alpha,\beta)}$ is a (3,4;q-1)-manifold for $0 < |\varepsilon| < \varepsilon^*$.

(ii) It is to be shown here first that $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ is regularly open, *i.e.*, that $\mathbb{B}^{\varepsilon-o}_{(\alpha,\beta)} = \mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$, for each $\varepsilon < 0$ with $|\varepsilon|$ sufficiently small. In view of (22), we must show that

$$\tilde{\mathbb{B}}^{\varepsilon^{o}}_{[\alpha,\beta]} = \mathbb{B}^{\varepsilon}_{(\alpha,\beta)} \quad \text{whenever } \varepsilon < 0, \quad |\varepsilon| \text{ sufficiently small.}$$

Let $\varepsilon < 0$. We begin by verifying that

$$\begin{aligned} \mathbb{B}^{\varepsilon} &:= \cup_{\zeta \in \mathbb{R}} \ \{\{Y \in \mathcal{B}^{o}_{\zeta} | \ \text{dist}\ (Y, \partial\mathcal{B}_{\zeta}) > -\varepsilon\} \times \{\zeta\}\} \\ &= \{\cup_{\zeta \in \mathbb{R}} \ \{\{Y \in \mathcal{B}^{o}_{\zeta} | \ \text{dist}\ (Y, \partial\mathcal{B}_{\zeta}) \geq -\varepsilon\} \times \{\zeta\}\}\}^{o} \qquad (25) \\ &= \tilde{\mathbb{B}}^{\varepsilon^{o}}. \end{aligned}$$

It has already been proven that $\mathbb{B}^{\varepsilon}$ is open, so the obvious inclusion $\mathbb{B}^{\varepsilon} \subset \tilde{\mathbb{B}}^{\varepsilon}$ allows us to assert that $\mathbb{B}^{\varepsilon} \subset \tilde{\mathbb{B}}^{\varepsilon^{o}}$. Assume next that $(X,t) \in \tilde{\mathbb{B}}^{\varepsilon^{o}}$, so $X \in \mathcal{B}^{o}_{t}$ with dist $(X, \partial\mathcal{B}_{t}) \geq -\varepsilon$, and there exist numbers $r \in (0,-2\varepsilon)$ and $\eta > 0$ such that $B^{3}_{r}(X) \times (t-\eta, t+\eta) \subset \tilde{\mathbb{B}}^{\varepsilon}$. Certainly, then,

$$B^{3}_{r}(X) \subset \{Y \in \mathcal{B}^{o}_{t} | \ \text{dist}\ (Y, \partial\mathcal{B}_{t}) \geq -\varepsilon\}. \qquad (26)$$

We wish to show that dist $(X, \partial\mathcal{B}_{t}) > -\varepsilon$; suppose dist $(X, \partial\mathcal{B}_{t}) = -\varepsilon$: there is some $\tilde{X} \in \partial\mathcal{B}_{t}$ with $|\tilde{X}-X|_{3} = -\varepsilon$, and the point $X + \frac{r}{2(-\varepsilon)}(\tilde{X}-X)$ clearly lies in $B^{3}_{r}(X)$. However,

$$\left|X+\frac{r}{2(-\varepsilon)}(\tilde{X}-X)-\tilde{X}\right|_3 = \left|\left(\frac{r}{2(-\varepsilon)}-1\right)(\tilde{X}-X)\right|_3$$

$$= \left|\frac{r}{2}+\varepsilon\right|$$

$$= -\varepsilon-\frac{r}{2}$$

$$< -\varepsilon,$$

which implies that $\operatorname{dist}\left(X+\frac{r}{2(-\varepsilon)}(\tilde{X}-X),\partial B_t\right) < -\varepsilon$, in violation of (26). Therefore, $\operatorname{dist}(X,\partial B_t) > -\varepsilon$, so $(X,t) \in \mathbb{B}^{\varepsilon}$. Equality (25) follows. Application of this result, supposing $|\varepsilon|$ is small enough that (22) holds, gives directly

$$\mathbb{B}^{\varepsilon-o}_{(\alpha,\beta)} = \tilde{\mathbb{B}}^{\varepsilon^o}_{[\alpha,\beta]}$$

$$= \{\tilde{\mathbb{B}}^{\varepsilon}\cap\{\mathbb{R}^3\times[\alpha,\beta]\}\}^o$$

$$= \tilde{\mathbb{B}}^{\varepsilon^o}\cap\{\mathbb{R}^3\times(\alpha,\beta)\}$$

$$= \mathbb{B}^{\varepsilon}\cap\{\mathbb{R}^3\times(\alpha,\beta)\}$$

$$= \mathbb{B}^{\varepsilon}_{(\alpha,\beta)}.$$

This proves our original assertion.

Choose $\alpha', \beta' \in \mathbb{R}$ with $\alpha' < \alpha$ and $\beta < \beta'$. Let $\varepsilon < 0$ be such that $|\varepsilon| < \varepsilon^*(M,\alpha',\beta')$ (cf., (24)), and each result obtained during the proof of (i) holds not only for α and β, but also when α and β are replaced by α' and β', respectively: we shall employ the criteria established in Remark [I.2.41.c] in order to show that the corresponding set $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ is a normal domain whenever $|\varepsilon|$ is small enough. Note first that $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)} \neq \emptyset$. In fact, by (24), we see that

$$\{Y \in B^{o}_{\zeta} | \text{ dist } (Y,\partial B_{\zeta}) > -\varepsilon\}\times\{\zeta\} \neq \emptyset \quad \text{for each} \quad \zeta \in [\alpha,\beta]. \tag{27}$$

One can show easily that $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ is bounded by using [V.10], while we have just seen that this set is regularly open. Moreover, the boundary of $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ can be written as

$$\partial\{\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}\} = \bar{\mathbb{B}}^{\varepsilon}_{[\alpha,\beta]} \cap \mathbb{B}^{\varepsilon'}_{(\alpha,\beta)} = \Gamma_0 \cup \Gamma_\alpha \cup \Gamma_\beta \cup S^{\varepsilon}_{(\alpha,\beta)},$$

wherein

$$\Gamma_0 := \{\{Y \in B^{o}_{\alpha} | \text{ dist } (Y,\partial B_{\alpha}) = -\varepsilon\}\times\{\alpha\}\}$$
$$\cup\{\{Y \in B^{o}_{\beta} | \text{ dist } (Y,\partial B_{\beta}) = -\varepsilon\}\times\{\beta\}\},$$

$$\Gamma_\alpha := \{Y \in B^{o}_{\alpha} | \text{ dist } (Y,\partial B_{\alpha}) > -\varepsilon\}\times\{\alpha\},$$

and

$$\Gamma_\beta := \{Y \in B^{o}_{\beta} | \text{ dist } (Y,\partial B_{\beta}) > -\varepsilon\}\times\{\beta\}.$$

It is, for the most part, routine to check that this decomposition fulfills the requirements of [I.2.41.c]. Γ_o is the union of two compact sets which are $(2,4;q-1)$-manifolds if $|\varepsilon|$ is sufficiently small, as we remarked prior to the proof of (i). Γ_α and Γ_β are non-void (cf., (27)), while it is trivial to see that each is relatively open in $\partial\{\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}\}$ as well as in $\mathbb{R}^3\times\{\alpha\}$ and $\mathbb{R}^3\times\{\beta\}$, respectively; from the latter fact, each is a $(3,4;\infty)$-manifold. Just as obvious are the inclusions $\Gamma_\alpha^- \subset \mathbb{R}^3\times\{\alpha\}$, $\Gamma_\beta^- \subset \mathbb{R}^3\times\{\beta\}$. From (i), it is known that $S^\varepsilon_{(\alpha,\beta)} = G^\varepsilon_{(\alpha,\beta)}((\partial\mathbb{B})_{(\alpha,\beta)})$ is a $(3,4;q-1)$-manifold. By its very definition, $S^\varepsilon_{(\alpha,\beta)}$ is open in $\partial\{\mathbb{B}^\varepsilon_{(\alpha,\beta)}\}$. Let us show that

$$S^{\varepsilon-}_{(\alpha,\beta)} = S^\varepsilon_{(\alpha,\beta)}\cup\{\{Y \in B^o_\alpha|\ \text{dist}\ (Y,\partial B_\alpha) = -\varepsilon\}\times\{\alpha\}\}$$
$$\cup\{\{Y \in B^o_\beta|\ \text{dist}\ (Y,\partial B_\beta) = -\varepsilon\}\times\{\beta\}\} \subset S^\varepsilon_{(\alpha',\beta')}. \tag{28}$$

Indeed, since $\alpha' < \alpha$ and $\beta < \beta'$, the inclusion claimed in (28) is clear from the results of the proof of (i), with α and β replaced by α' and β', respectively, for (8) then gives

$$S^\varepsilon_{(\alpha',\beta')} = \cup_{\alpha'<\zeta<\beta'}\ \{\{Y \in B^o_\zeta|\ \text{dist}\ (Y,\partial B_\zeta) = -\varepsilon\}\times\{\zeta\}\}.$$

To prove the equality in (28), suppose first that $X \in B^o_\alpha$ with $\text{dist}\ (X,\partial B_\alpha) = -\varepsilon$. There exists $\tilde{X} \in \partial B_\alpha$ such that $X = \tilde{X}+\varepsilon\nu(\tilde{X},\alpha)$. Using the reference pair (B_α,X^α) for M, employed in the proof of (i), and selecting a sequence $(s_n)^\infty_{n=1}$ in (α,β) converging to α, the sequence $((X^\alpha(\tilde{X},s_n)+\varepsilon\nu(X^\alpha(\tilde{X},s_n),s_n),s_n)^\infty_{n=1}$ lies in $S^\varepsilon_{(\alpha,\beta)}$ and converges to (X,α), showing that this point is in $S^{\varepsilon-}_{(\alpha,\beta)}$. We arrive at the same conclusion in case $X \in B^o_\beta$ with $\text{dist}\ (X,\partial B_\beta) = -\varepsilon$. Thus, the union appearing in (28) is contained in $S^{\varepsilon-}_{(\alpha,\beta)}$. On the other hand, suppose $(X,t) \in S^{\varepsilon-}_{(\alpha,\beta)}$: there exists a sequence $((X_n,t_n))^\infty_{n=1} \subset S^\varepsilon_{(\alpha,\beta)}$ converging to (X,t), so $\lim_{n\to\infty} X_n = X$, $\lim_{n\to\infty} t_n = t$, and $X_n \in B^o_{t_n}$ with dist $(X_n, \partial\mathcal{B}_{t_n}) = -\varepsilon$ for each $n \in \mathbb{N}$. Since $\{t_n\}_{n=1}^{\infty} \subset (\alpha,\beta)$, we have $t \in [\alpha,\beta]$. We wish to show that dist $(X, \partial\mathcal{B}_t) = -\varepsilon$. For each $n \in \mathbb{N}$, there exists $\tilde{X}_n \in \partial\mathcal{B}_{t_n}$ for which $X_n = \tilde{X}_n + \varepsilon\nu(\tilde{X}_n, t_n)$; in fact, it is clear that $(\tilde{X}_n, t_n) = G^{\varepsilon^{-1}}_{(\alpha',\beta')}(X_n, t_n)$, whence it follows that $(\tilde{X}_n)_{n=1}^{\infty}$ converges to a point $\tilde{X} \in \partial\mathcal{B}_t$ and $(\tilde{X}, t) = G^{\varepsilon^{-1}}_{(\alpha',\beta')}(X,t)$, or $(X,t) = G^{\varepsilon}_{(\alpha',\beta')}(\tilde{X},t)$. In view of (24) (written for α', β'), it follows that dist $(X, \partial\mathcal{B}_t) = -\varepsilon$. Consequently, (X,t) is in $\cup_{\alpha \le \zeta \le \beta} \{\{Y \in \mathcal{B}^o_\zeta |$ dist $(Y, \partial\mathcal{B}_\zeta) = -\varepsilon\} \times \{\zeta\}\}$, which is just the union appearing in (28) (cf., (8)). Having completed the proof of (28), we can assert that $S^{\varepsilon-}_{(\alpha,\beta)}$ is contained in the $(3,4;q-1)$-manifold $S^{\varepsilon}_{(\alpha',\beta')}$. Finally, since reasoning as in [VI.68] implies that

$$\Gamma^-_\alpha = \{Y \in \mathcal{B}^o_\alpha | \text{ dist } (Y, \partial\mathcal{B}_\alpha) \ge -\varepsilon\} \times \{\alpha\},$$

and

$$\Gamma^-_\beta = \{Y \in \mathcal{B}^o_\beta | \text{ dist } (Y, \partial\mathcal{B}_\beta) \ge -\varepsilon\} \times \{\beta\},$$

provided $|\varepsilon|$ is sufficiently small, we can also use (28) to conclude that $\Gamma^-_\alpha \cap S^{\varepsilon-}_{(\alpha,\beta)} \subset \Gamma_0$ and $\Gamma^-_\beta \cap S^{\varepsilon-}_{(\alpha,\beta)} \subset \Gamma_0$.

Having checked that all requirements of [I.2.41.c] are fulfilled by $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ if $\varepsilon < 0$ and $|\varepsilon|$ is sufficiently small, the set is a normal domain for such ε.

(iii) By proceeding as in the proof of (ii), *mutatis* *mutandis*, one can show that $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ is a regularly open subset of $\mathbb{R}^4$ whenever ε is positive and small enough that (22) holds; we shall suppose that this has been done.

Now, select $\tilde{c} > c^*$, and fix any positive ρ so large that $B^3_{\tilde{c}(\beta-\alpha)}(\mathcal{B}_\alpha) \subset B^3_{\rho/2}(0)$. Obviously, $\mathcal{B}_\alpha \subset B^3_{\rho/2}(0)$. But also, for each $\zeta \in (\alpha,\beta]$, by recalling [V.10], we find that

$$\mathcal{B}_\zeta = \mathcal{B}_{\alpha+(\zeta-\alpha)} \subset B^3_{\tilde{c}(\zeta-\alpha)}(\mathcal{B}_\alpha) \subset B^3_{\tilde{c}(\beta-\alpha)}(\mathcal{B}_\alpha) \subset B^3_{\rho/2}(0).$$

Then it is easy to show that for each $\sigma \in (0, \rho/2]$,

$$\{Y \in \mathbb{R}^3 \mid \text{dist}\,(Y,\mathcal{B}_\zeta) \le \sigma\} \subset B^3_\rho(0) \quad \text{whenever} \quad \zeta \in [\alpha,\beta], \tag{29}$$

hence that

$$\begin{gathered}\{Y \in \mathcal{B}'_\zeta \mid \text{dist}\,(Y,\partial\mathcal{B}_\zeta) > \sigma\} \cap B^3_\rho(0) \quad \text{is non-void and open} \\ \text{in} \quad \mathbb{R}^3 \quad \text{for each} \quad \zeta \in [\alpha,\beta].\end{gathered} \tag{30}$$

Thus, defining

$$\mathbb{B}^{\varepsilon,\rho}_{(\alpha,\beta)} := \mathbb{B}^{\varepsilon}_{(\alpha,\beta)} \cap \{B^3_\rho(0) \times (\alpha,\beta)\} \quad \text{for} \quad \varepsilon \in (0, \rho/2),$$

we obtain for each such ε a non-void open subset of $\mathbb{R}^4$ possessing a non-void ζ-section which is relatively open in $\mathbb{R}^3 \times \{\zeta\}$, for each $\zeta \in [\alpha,\beta]$. We wish to show that whenever ε is sufficiently small the corresponding $\mathbb{B}^{\varepsilon,\rho}_{(\alpha,\beta)}$ is a normal domain; for this, we shall rely once again upon the result of [I.2.41.c]. Let us first satisfy ourselves that

$$\mathbb{B}^{\varepsilon,\rho^-}_{(\alpha,\beta)} = \mathbb{B}^{\varepsilon-}_{(\alpha,\beta)} \cap \{B^3_\rho(0)^- \times [\alpha,\beta]\} \quad \text{for each sufficiently small positive } \varepsilon. \tag{31}$$

The inclusion $\mathbb{B}^{\varepsilon,\rho^-}_{(\alpha,\beta)} \subset \mathbb{B}^{\varepsilon-}_{(\alpha,\beta)} \cap \{B^3_\rho(0)^- \times [\alpha,\beta]\}$ is plain enough, since the latter set is closed and contains $\mathbb{B}^{\varepsilon,\rho}_{(\alpha,\beta)}$. Suppose ε is so small that (22) holds. Then

$$\begin{aligned} &\mathbb{B}^{\varepsilon-}_{(\alpha,\beta)} \cap \{B^3_\rho(0)^- \times [\alpha,\beta]\} \\ &= \cup_{\alpha \le \zeta \le \beta} \{\{Y \in \mathcal{B}'_\zeta \mid \text{dist}\,(Y, \partial\mathcal{B}_\zeta) \ge \varepsilon, \quad |Y|_3 \le \rho\} \times \{\zeta\}\}. \end{aligned} \tag{32}$$

In fact, if we choose $\tilde{\varepsilon} \in (0, \min\{\varepsilon^*, \rho/2\})$ and assume that $0 < \varepsilon < \tilde{\varepsilon}$, then we have not only (32), but also

$$\{Y \in \mathcal{B}'_\zeta \mid \text{dist}\,(Y, \partial\mathcal{B}_\zeta) = \sigma\} \subset B^3_\rho(0) \quad \text{for each} \quad \sigma \in (0, \tilde{\varepsilon}) \quad \text{and} \quad \zeta \in [\alpha,\beta], \tag{33}$$

$$\{Y \in \mathcal{B}'_\zeta \mid \text{dist}\,(Y, \partial\mathcal{B}_\zeta) > \varepsilon\} \cap B^3_\rho(0) \quad \text{is non-void and open in} \quad \mathbb{R}^3 \quad \text{for each} \quad \zeta \in [\alpha,\beta], \tag{34}$$

and

$$G^\sigma(\partial\mathcal{B}_\zeta \times \{\zeta\}) = \{Y \in \mathcal{B}'_\zeta \mid \text{dist}\,(Y, \partial\mathcal{B}_\zeta) = \sigma\} \times \{\zeta\} \quad \text{if} \quad \sigma \in (0, \tilde{\varepsilon}) \quad \text{and} \quad \zeta \in [\alpha,\beta], \tag{35}$$

following from (29), (30), and (22), respectively. Now, choose $(X,t) \in \mathbb{B}^{\varepsilon-}_{(\alpha,\beta)} \cap \{B^3_\rho(0)^- \times [\alpha,\beta]\}$, *i.e.*, by (32), such that $t \in [\alpha,\beta]$, $X \in \mathcal{B}'_t \cap B^3_\rho(0)^-$, and $\text{dist}\,(X, \partial\mathcal{B}_t) \ge \varepsilon$: then one can construct a sequence in $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ which converges to (X,t). The proof of this fact is most easily accomplished by using (33)-(35) and proceeding essentially as in the proof of (22), considering each of the cases $t \in (\alpha,\beta)$, $t = \alpha$, and $t = \beta$. The additional subcases which must be examined here, because of the presence of the bounding set $\partial B^3_\rho(0)\times[\alpha,\beta]$, can be handled in an obvious manner, with (34). We shall omit the details of the verification. We conclude that $(X,t) \in \mathbb{B}^{\varepsilon-}_{(\alpha,\beta)}$, and so (31) is true whenever $0 < \varepsilon < \min\{\tilde{\varepsilon}, \rho/2\}$.

Now, again if $0 < \tilde{\varepsilon} < \min\{\tilde{\varepsilon}, \rho/2\}$, since $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ is regularly open and (31) holds, we find immediately that $\mathbb{B}^{\varepsilon,\rho}_{(\alpha,\beta)}$ is also regularly open:

$$\begin{aligned}
\mathbb{B}^{\varepsilon,\rho^{-o}}_{(\alpha,\beta)} &= \{\mathbb{B}^{\varepsilon-}_{(\alpha,\beta)}\cap\{B^3_\rho(0)^{-}\times[\alpha,\beta]\}\}^{o} \\
&= \mathbb{B}^{\varepsilon^{-o}}_{(\alpha,\beta)}\cap\{B^3_\rho(0)\times(\alpha,\beta)\} \\
&= \mathbb{B}^{\varepsilon}_{(\alpha,\beta)}\cap\{B^3_\rho(0)\times(\alpha,\beta)\} \\
&= \mathbb{B}^{\varepsilon,\rho}_{(\alpha,\beta)}.
\end{aligned}$$

Choose $\alpha', \beta' \in \mathbb{R}$ with $\alpha' < \alpha$ and $\beta < \beta'$. Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\tilde{\varepsilon}, \rho/2, \varepsilon^*(M,\alpha',\beta')\}$ and each result obtained during the proof of (i) holds for α, β and with α', β' replacing α, β, respectively. Then $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ is a non-void, bounded, regularly open subset of $\mathbb{R}^4$. Moreover, it is easy to identify a decomposition of $\partial\{\mathbb{B}^{\varepsilon,\rho}_{(\alpha,\beta)}\} = \mathbb{B}^{\varepsilon,\rho^-}_{(\alpha,\beta)}\cap\mathbb{B}^{\varepsilon,\rho'}_{(\alpha,\beta)}$ which fulfills all requirements set forth in [I.2.41.c] if ε is sufficiently small;

the details can be supplied by basing the reasoning on the proof carried out in (ii). Thus, $\mathbb{B}^{\varepsilon,\rho}_{(\alpha,\beta)}$ is a normal domain whenever ε is sufficiently small and positive.

(iv) We suppose $|\varepsilon|$ to be so small that the statements proven in (i)-(iii) hold. Fix $(Z,\zeta) \in (\partial\mathbb{B})_{(\alpha,\beta)}$. We shall first show that

$$T_{S^{\varepsilon}_{(\alpha,\beta)}}(G^{\varepsilon}_{(\alpha,\beta)}(Z,\zeta)) = T_{\partial\mathbb{B}}(Z,\zeta) \quad \text{for} \quad |\varepsilon| \text{ sufficiently small,} \tag{36}$$

for which it suffices to secure the inclusion

$$T_{S^{\varepsilon}_{(\alpha,\beta)}}(G^{\varepsilon}_{(\alpha,\beta)}(Z,\zeta)) \subset T_{\partial\mathbb{B}}(Z,\zeta) \quad \text{if} \quad |\varepsilon| \text{ is small enough.} \tag{37}$$

To prove (37), let $\beta \in T_{S^{\varepsilon}_{(\alpha,\beta)}}(G^{\varepsilon}_{(\alpha,\beta)}(Z,\zeta))$. There exist $\delta > 0$ and $\psi^{\varepsilon} \in C^1((-\delta,\delta);\mathbb{R}^4)$ with $\psi^{\varepsilon}(-\delta,\delta) \subset S^{\varepsilon}_{(\alpha,\beta)}$, $\psi^{\varepsilon}(0) = G^{\varepsilon}_{(\alpha,\beta)}(Z,\zeta)$, and $\psi^{\varepsilon\prime}(0) = \beta$. Define $\psi\colon (-\delta,\delta) \to \mathbb{R}^4$ by

$$\psi(\sigma) := G^{\varepsilon^{-1}}_{(\alpha,\beta)} \circ \psi^{\varepsilon}(\sigma) \quad \text{for} \quad |\sigma| < \delta, \tag{38}$$

and $f_\nu\colon (-\delta,\delta) \to \mathbb{R}^4$ according to

$$f_\nu(\sigma) := (\nu \circ \psi(\sigma), 0) \quad \text{for} \quad |\sigma| < \delta. \tag{39}$$

Let us assume, for the moment, that

$$\psi|(-\delta_0,\delta_0) \in C^1((-\delta_0,\delta_0);\mathbb{R}^4) \quad \text{for some} \quad \delta_0 \in (0,\delta]. \tag{40}$$

Then, since $\psi((-\delta_0,\delta_0)) \subset \partial\mathbb{B}$ and $\psi(0) = (Z,\zeta)$, it follows that

$\psi'(0) \in T_{\partial\mathbb{B}}(Z,\zeta)$. Further, recalling that $\mathbb{B}^o$ is a q-regular domain ([V.47.iii.2]), there exist an open neighborhood of (Z,ζ) in $\mathbb{R}^4$, $U_{(Z,\zeta)}$, and a function $\Phi_{(Z,\zeta)} \in C^q(U_{(Z,\zeta)})$ such that grad $\Phi_{(Z,\zeta)}$ does not vanish in $U_{(Z,\zeta)}$, and

$$\begin{aligned}\nu_{\partial\mathbb{B}}(Y,s) &= \{1+\upsilon^2(Y,s)\}^{-1/2}\cdot(\nu(Y,s),-\upsilon(Y,s))\\ &= |\mathrm{grad}\ \Phi_{(Z,\zeta)}(Y,s)|_4^{-1}\cdot\ \mathrm{grad}\ \Phi_{(Z,\zeta)}(Y,s) \qquad (41)\\ &\text{for each} \quad (Y,s) \in \partial\mathbb{B}\cap U_{(Z,\zeta)}.\end{aligned}$$

Thus, for the function $\mathrm{grad}^\wedge\ \Phi_{(Z,\zeta)} := (\Phi_{(Z,\zeta),1}, \Phi_{(Z,\zeta),2}, \Phi_{(Z,\zeta),3}) \in C^{q-1}(U_{(Z,\zeta)};\mathbb{R}^3)$, (41) evidently implies that

$$\begin{aligned}\nu(Y,s) &= |\mathrm{grad}^\wedge\ \Phi_{(Z,\zeta)}(Y,s)|_3^{-1}\cdot\ \mathrm{grad}^\wedge\ \Phi_{(Z,\zeta)}(Y,s) \qquad (42)\\ &\text{for each} \quad (Y,s) \in \partial\mathbb{B}\cap U_{(Z,\zeta)}.\end{aligned}$$

In view of the definition (39), (40) and (42) together imply that f_ν is of class C^1 on a neighborhood of 0. Since $|f_\nu(\sigma)|_4^2 = 1$ for $|\sigma| < \delta$, we find that $f_\nu(0)\bullet f'_\nu(0) = 0$, whence

$$\begin{aligned}\nu_{\partial\mathbb{B}}(Z,\zeta)\bullet f'_\nu(0) &= \{1+\upsilon^2(Z,\zeta)\}^{-1/2}(\nu(Z,\zeta),-\upsilon(Z,\zeta))\bullet((\nu\circ\psi)'(0),0)\\ &= \{1+\upsilon^2(Z,\zeta)\}^{-1/2}f_\nu(0)\bullet f'_\nu(0)\\ &= 0,\end{aligned}$$

i.e., $f'_\nu(0) \in T_{\partial\mathbb{B}}(Z,\zeta)$. But then, observing that

$$\psi^\varepsilon = G^\varepsilon_{(\alpha,\beta)}\circ G^{\varepsilon^{-1}}_{(\alpha,\beta)}\circ\psi^\varepsilon = G^\varepsilon_{(\alpha,\beta)}\circ\psi = \psi+\varepsilon f_\nu \quad \text{on} \quad (-\delta,\delta), \qquad (43)$$

the desired result

$$\beta = \psi^{\varepsilon'}(0) = \psi'(0)+\varepsilon f_\nu'(0) \in T_{\partial\mathbb{B}}(Z,\zeta)$$

follows. This implies that (37), hence also (36), is correct, provided (40) obtains.

Let us check, then, that (40) is true for all ε with $|\varepsilon|$ sufficiently small, independently of the point (Z,ζ) chosen in $(\partial\mathbb{B})_{(\alpha,\beta)}$. First, by the q-regularity of $\mathbb{B}^o$ and the compactness of $(\partial\mathbb{B})_{[\alpha,\beta]}$, we can find a finite family of open subsets of $\mathbb{R}^4$ forming a cover for $(\partial\mathbb{B})_{[\alpha,\beta]}$, $\{U_i\}_{i=1}^p$, and a corresponding collection of functions $\{\Phi_i \in C^q(U_i)\}_{i=1}^p$ such that, for each $i \in \{1,\dots,p\}$, for certain positive numbers m_i and M_i,

$$m_i \leq |\text{grad}^\wedge\ \Phi_i|_3 \leq M_i \quad \text{on} \quad U_i, \tag{44}$$

$$|\Phi_{i,jk}| \leq M_i \quad \text{on} \quad U_i \quad \text{for} \quad j,k = 1,2,3, \text{ and } 4, \tag{45}$$

and

$$\nu(Y,s) = |\text{grad}^\wedge\ \Phi_{\underline{i}}(Y,s)|_3^{-1}\cdot \text{grad}^\wedge\ \Phi_{\underline{i}}(Y,s) \quad \text{for each} \quad (Y,s) \in \partial\mathbb{B}\cap U_i, \tag{46}$$

wherein $\text{grad}^\wedge\ \Phi_i := (\Phi_{i,1}, \Phi_{i,2}, \Phi_{i,3}) \in C^{q-1}(U_i;\mathbb{R}^3)$; cf., the reasoning accompanying (41) and (42). Because of (44) and (45), we have

$$\det\begin{pmatrix} \left(\delta_{jk}+\varepsilon\left(\dfrac{\Phi_{i,j}}{|\mathrm{grad}\hat{}\ \Phi_{i}|_3}\right)_{,k}(Y,s)\right)_{\substack{1\le j\le 3\\ 1\le k\le 4}} & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{pmatrix} > 0 \tag{47}$$

whenever $|\varepsilon|$ is sufficiently small,
for each $(Y,s) \in \partial\mathbb{B}\cap U_i$, for
each $i \in \{1,\ldots,p\}$.

Now, choose any ε fulfilling the latter requirement (and the restrictions previously imposed). Choose $\ell \in \{1,\ldots,p\}$ such that $(Z,\zeta) \in U_\ell$, and define $F\colon\ U_\ell\times(-\delta,\delta) \to \mathbb{R}^4$ by

$$F(Y,s,\sigma) := (Y,s)+\varepsilon\left(\frac{\mathrm{grad}\hat{}\ \Phi_\ell(Y,s)}{|\mathrm{grad}\hat{}\ \Phi_\ell(Y,s)|_3},0\right)-\psi^\varepsilon(\sigma) \tag{48}$$

for $(Y,s) \in U_\ell$, $|\sigma| < \delta$;

it is clear that $F \in C^1(U_\ell\times(-\delta,\delta);\mathbb{R}^4)$ (recalling that $\psi^\varepsilon \in C^1((-\delta,\delta);\mathbb{R}^4)$). Since (43) gives

$$\psi(\sigma)+\varepsilon(\nu\circ\psi(\sigma),0)-\psi^\varepsilon(\sigma) = 0 \quad \text{for} \quad |\sigma| < \delta, \tag{49}$$

with (46) we find

$$F(\psi(\sigma),\sigma) = \psi(\sigma)+\varepsilon(\nu(\psi(\sigma)),0)-\psi^\varepsilon(\sigma) = 0 \quad \text{for} \quad |\sigma| < \delta, \tag{50}$$

so, in particular,

$$F(Z,\zeta,0) = F(\psi(0),0) = 0. \tag{51}$$

Moreover, by (47), it is easy to see that

$$\det\left((F^j_{,k}(Z,\zeta,0))_{1\leq j,k\leq 4}\right) > 0. \tag{52}$$

Having (51) and (52), we can apply the implicit function theorem to F and the point $(Z,\zeta,0)$, and combine the properties of the resultant implicitly defined function $\tilde{\psi} \in C^1((-\tilde{\delta},\tilde{\delta});\mathbb{R}^4)$ with (50) and the continuity of ψ to deduce that ψ and $\tilde{\psi}$ must coincide in some neighborhood of 0. The details of this reasoning differ in no essential from those laid out in [VI.68], in the proof of [I.2.43]. Clearly, (40) now follows.

Maintaining the restrictions placed on $|\varepsilon|$, from (36) we can conclude that

$$N_{S^{\varepsilon}_{(\alpha,\beta)}}(G^{\varepsilon}(Z,\zeta)) = N_{\partial\mathbb{B}}(Z,\zeta). \tag{53}$$

If $\varepsilon > 0$, $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ is a normal domain, while if $\varepsilon < 0$ and ρ is chosen as in (iii), then $\mathbb{B}^{\varepsilon,\rho}_{(\alpha,\beta)}$ is a normal domain; in either case the exterior unit normal field for $S^{\varepsilon}_{(\alpha,\beta)}$, $\nu^{\varepsilon}_{(\alpha,\beta)} := \nu_{S^{\varepsilon}_{(\alpha,\beta)}}$, is defined. The result (53) implies that $\nu^{\varepsilon}_{(\alpha,\beta)}(G^{\varepsilon}_{(\alpha,\beta)}(Z,\zeta))$ must be given by one of $\nu_{\partial\mathbb{B}}(Z,\zeta)$, $-\nu_{\partial\mathbb{B}}(Z,\zeta)$. Consider the case in which $\varepsilon < 0$. It is easy to see that the set

$$(\mathcal{B}^{o}_{s})_{\varepsilon} := \{Y \in \mathcal{B}^{o}_{s} |\ \operatorname{dist}(Y,\partial\mathcal{B}_{s}) > -\varepsilon\}$$

is a $(q-1)$-regular domain in $\mathbb{R}^3$, with exterior unit normal field given by

$$\nu_{\partial(\mathcal{B}_s^o)_\varepsilon} = \nu(G^{\varepsilon^{-1}}_{(\alpha,\beta)}(\cdot,s)) \tag{54}$$

whenever $|\varepsilon|$ is sufficiently small and $s \in (\alpha,\beta)$. In fact, (54) follows by arguing as in the proof of [VI.68.iii], noting that $Y+\sigma\nu(Y,s) \in \mathcal{B}_s^o$ with $\text{dist}\ (Y+\sigma\nu(Y,s),\partial\mathcal{B}_s) = -\sigma$ whenever $|\sigma|$ is sufficiently small, $s \in (\alpha,\beta)$, and $Y \in \partial\mathcal{B}_s$. On the other hand, we can also show that

$$\nu_{\partial(\mathcal{B}_s^o)_\varepsilon} = \hat{\nu}^{\varepsilon}_{(\alpha,\beta)}(\cdot,s) \quad \text{for each} \quad s \in (\alpha,\beta), \tag{55}$$

wherein $\hat{\nu}^{\varepsilon}_{(\alpha,\beta)}$ denotes the field of unit magnitude constructed from $(\nu^{\varepsilon 1}_{(\alpha,\beta)}, \nu^{\varepsilon 2}_{(\alpha,\beta)}, \nu^{\varepsilon 3}_{(\alpha,\beta)})$. Indeed, by choosing $s \in (\alpha,\beta)$ and $Y \in \mathcal{B}_s^o$ with $\text{dist}\ (Y,\partial\mathcal{B}_s) = -\varepsilon$, so $(Y,s) \in S^{\varepsilon}_{(\alpha,\beta)}$, then the $(q-1)$-regularity of $\mathbb{B}^{\varepsilon}_{(\alpha,\beta)}$ at (Y,s) shows that there exist an open neighborhood of (Y,s) in $\mathbb{R}^4$, $U^{\varepsilon}_{(Y,s)}$, and a function $\phi^{\varepsilon}_{(Y,s)} \in C^{q-1}(U^{\varepsilon}_{(Y,s)})$ with the properties described by [I.2.27]. Thus, by Remark [I.2.32.b],

$$\nu^{\varepsilon}_{(\alpha,\beta)}(Y,s) = |\text{grad}\ \phi^{\varepsilon}_{(Y,s)}(Y,s)|_4^{-1} \cdot \text{grad}\ \phi^{\varepsilon}_{(Y,s)}(Y,s). \tag{56}$$

Since $\nu^{\varepsilon}_{(\alpha,\beta)}(Y,s)$ is a non-zero multiple of $\nu_{\partial\mathbb{B}}(G^{\varepsilon^{-1}}_{(\alpha,\beta)}(Y,s))$, we see that $(\phi^{\varepsilon}_{(Y,s),1}(Y,s), \phi^{\varepsilon}_{(Y,s),2}(Y,s), \phi^{\varepsilon}_{(Y,s),3}(Y,s)) \neq 0$, so it can be assumed that $(\phi^{\varepsilon}_{(Y,s),1}, \phi^{\varepsilon}_{(Y,s),2}, \phi^{\varepsilon}_{(Y,s),3})$ does not vanish in $U^{\varepsilon}_{(Y,s)}$. But then it is clear that the open neighborhood of Y in $\mathbb{R}^3$ given by $U^{\varepsilon}_Y := \{\tilde{Y} \in \mathbb{R}^3 \mid (\tilde{Y},s) \in U^{\varepsilon}_{(Y,s)}\}$ and the function $\tilde{Y} \mapsto \phi^{\varepsilon}_{(Y,s)}(\tilde{Y},s)$ in $C^{q-1}(U^{\varepsilon}_Y)$ possess the properties described in [I.2.27] relative to the $(q-1)$-regular domain

$(B^o_s)_\varepsilon \subset \mathbb{R}^3$ and the point $Y \in \partial(B^o_s)_\varepsilon$. This, in turn, implies that

$$\nu_{\partial(B^o_s)_\varepsilon}(Y) = |\text{grad}^\wedge\ \Phi^\varepsilon_{(Y,s)}(Y,s)|_3^{-1} \cdot \text{grad}^\wedge\ \Phi^\varepsilon_{(Y,s)}(Y,s), \tag{57}$$

wherein $\text{grad}^\wedge\ \Phi^\varepsilon_{(Y,s)} := (\Phi^\varepsilon_{(Y,s),1}, \Phi^\varepsilon_{(Y,s),2}, \Phi^\varepsilon_{(Y,s),3})$. Now (55) follows from (56) and (57). Finally, upon comparing (54) and (55), keeping in mind the relation

$$\nu_{\partial\mathbb{B}} = \{1+\upsilon^2\}^{-1/2} \cdot (\nu, -\upsilon),$$

we infer that

$$\nu^\varepsilon_{(\alpha,\beta)}(G^\varepsilon_{(\alpha,\beta)}(Z,\zeta)) = \nu_{\partial\mathbb{B}}(Z,\zeta) \quad \text{for each} \quad (Z,\zeta) \in (\partial\mathbb{B})_{(\alpha,\beta)},$$
$$\text{whenever} \quad \varepsilon < 0 \quad \text{and} \quad |\varepsilon| \quad \text{is sufficiently small.} \tag{58}$$

Analogous reasoning leads to the equality

$$\nu^\varepsilon_{(\alpha,\beta)}(G^\varepsilon_{(\alpha,\beta)}(Z,\zeta)) = -\nu_{\partial\mathbb{B}}(Z,\zeta) \quad \text{for each} \quad (Z,\zeta) \in (\partial\mathbb{B})_{(\alpha,\beta)},$$
$$\text{whenever} \quad \varepsilon \quad \text{is positive and sufficiently small.} \tag{59}$$

The assertion of (iv) is a consequence of (58) and (59).

(v) This statement has been established in the proof of (i). $\square$.

V.A. APPENDIX

A TOPOLOGICAL RESULT

The following topological fact implies the validity of Lemma [V.6]:

L E M M A. *Let* T_1 *be a topological space,* T_2 *a first-countable Hausdorff space, and* $f\colon T_1 \to T_2$ *a continuous bijection with the following property:*

whenever K_2 *is a compact subset of* T_2,
there exists a compact $K_1 \subset T_1$ *such*
that $K_2 \subset f(K_1)$.

Then f^{-1} *is continuous, so* f *is a homeomorphism.*

P R O O F. In this setting, it suffices to show that $f^{-1}\colon T_2 \to T_1$ is sequentially continuous. Accordingly, let $(x_i)_{i=1}^{\infty}$ be a sequence in T_2 converging to $x_0 \in T_2$. The set $K_2 := \{x_i \mid i = 0,1,\dots\}$ is compact in T_2, so there exists a compact $K_1 \subset T_1$ such that $K_2 \subset f(K_1)$. Now, $f_1 := f|K_1\colon K_1 \to f(K_1)$ is bijective and continuous when K_1 and $f(K_1)$ are equipped with the respective relative topologies; since K_1 is compact and $f(K_1)$ is Hausdorff, f_1 is a homeomorphism, so $f_1^{-1}\colon f(K_1) \to K_1$ is continuous. Clearly, $(x_i)_{i=1}^{\infty}$ converges to x_0 in $f(K_1)$, whence

$$\lim_{i \to \infty} f^{-1}(x_i) = \lim_{i \to \infty} f_1^{-1}(x_i) = f_1^{-1}(x_0) = f^{-1}(x_0) \quad \text{in} \quad K_1,$$

which implies, in turn,

$$\lim_{i \to \infty} f^{-1}(x_i) = f^{-1}(x_0) \quad \text{in} \quad \mathcal{T}_1,$$

as well. Thus, f^{-1} is sequentially continuous, and so also continuous, since $\mathcal{T}_2$ is a first-countable space. $\square$.

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